Basis

Definition

If V is a vector space and S is a subset of V then S is a basis for V if

1. S is linearly independent and
2. S spans V.

Theorem

If S is a basis for V then every vector $v \in V$ can be expressed in the form

$$v = k_1 v_1 + k_2 v_2 + \ldots + k_n v_n$$

for distinct $v_1, \ldots, v_n \in S$ in exactly one way.

Dimension

Definition

We say that a vector space is finite-dimensional if it has a finite basis.

Theorem

If V is a vector space with a basis of n vectors then

1. *any subset of V with more than n vectors is linearly dependent, and*
2. *any subset of V with fewer than n vectors does not span V.*

Corollary

If V is a finite-dimensional vector space then all bases for V have the same size; we call this size the dimension of V.

- Suppose that $v_1, \ldots, v_k$ are vectors in $\mathbb{R}^n$; the big question is, what is the span of these vectors?
- Consider the matrix A with rows consisting of $v_1, \ldots, v_k$; A is $k \times n$.
- Now let $B = rref(A)$, the reduced row echelon form of A. We claim two things:
- First, the rows of B still span the same subspace as $v_1, \ldots, v_k$ and
- second, the non-zero rows form a basis for this subspace.
- In general, the subspace spanned by the rows of a matrix is called the row space and if the matrix is $k \times n$ then the row space is a subspace of $\mathbb{R}^n$. The dimension of the row-space of A is called the rank of A.

Example

- $$A = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & -1 \\ 1 & 1 & 1 & 0 & 3 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 3 & 0 \end{pmatrix}$$
- $$rref(A) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Plus/Minus Theorem

Theorem (Plus/Minus Theorem, 4.5.3)

Suppose that S is a non-empty subset of a vector space V. Then

1. *if S is linearly independent and $v \in V$ is not in span(S) then $S \cup \{v\}$ is linearly independent, and*
2. *if $v \in S$ can be expressed as a linear combination of vectors from $S \setminus \{v\}$ then span$(S) =$ span$(S \setminus \{v\})$.*

Corollaries

Corollary

Suppose that V is an n-dimensional vector space and S is a subset of V.

1. *If S spans V then S contains a basis for V.*
2. *If S is linearly independent then S is contained in a basis for V.*
3. *If S contains exactly n vectors then S is a basis for V iff S is linearly independent.*

Corollary

If W is a subspace of a finite-dimensional vector space V then

1. *W is finite-dimensional;*
2. *$\dim(W) \leq \dim(V)$; and*
3. *$\dim(W) = \dim(V)$ iff $W = V$.*

The column-space of a matrix

- We return to the problem of subspaces of $\mathbb{R}^n$. The question is: given vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$, how do we find a basis for the subspace W spanned by these vectors *from among these vectors.*
- Consider a matrix A formed by placing $v_1, \ldots, v_k$ in the columns; A is $n \times k$; let $B = rref(A)$, the reduced row-echelon form of A.
- The claim is that the vectors in A which correspond to the columns of B with leading 1's form a basis for W.
- Notice that this means that the dimension of W is the same as the rank of A.
- In general, for a matrix A which is $k \times n$, the subspace generated by the columns is called the column space of A, a subspace of $\mathbb{R}^k$, and its dimension is the same as the rank of A.