Subspaces of $\mathbb{R}^n$

- Suppose that $v_1, \ldots, v_k$ are vectors in $\mathbb{R}^n$; the big question is, what is the span of these vectors?
- Consider the matrix A with rows consisting of $v_1, \ldots, v_k$; A is $k \times n$.
- Now let $B = rref(A)$, the reduced row echelon form of A. We claim two things:
- First, the rows of B still span the same subspace as $v_1, \ldots, v_k$ and
- second, the non-zero rows form a basis for this subspace.
- In general, the subspace spanned by the rows of a matrix is called the row space and if the matrix is $k \times n$ then the row space is a subspace of $\mathbb{R}^n$. The dimension of the row-space of A is called the rank of A.

Example

- $$A = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & -1 \\ 1 & 1 & 1 & 0 & 3 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 3 & 0 \end{pmatrix}$$
- $$rref(A) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The nullspace and the nullity

- Remember that the nullspace for an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ consisting of all x such that $Ax = 0$.
- The nullity of A, nullity(A), is the dimension of the nullspace of A.

Theorem (Dimension Theorem for Matrices, 4.8.2)

For an $m \times n$ matrix A, rank(A) + nullity(A) = n

The column-space of a matrix

- We return to the problem of subspaces of $\mathbb{R}^n$. The question is: given vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$, how do we find a basis for the subspace W spanned by these vectors *from among these vectors.*
- Consider a matrix A formed by placing $v_1, \ldots, v_k$ in the columns; A is $n \times k$; let $B = rref(A)$, the reduced row-echelon form of A.
- The claim is that the vectors in A which correspond to the columns of B with leading 1's form a basis for W.
- Notice that this means that the dimension of W is the same as the rank of A.
- In general, for a matrix A which is $k \times n$, the subspace generated by the columns is called the column space of A, a subspace of $\mathbb{R}^k$, and its dimension is the same as the rank of A.

Back to the example

- $$A = \begin{pmatrix} 1 & 1 & 1 & 1; \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 \\ 3 & 3 & 1 & 3 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$
- $$rref(A) = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$