Mathematics 2R3 Test 3

Dr. Hart

Nov. 27, 2019

Name: Solutions.

Student No.:

- The test is 50 minutes long.
- The test has 6 pages and 5 questions and is printed on BOTH sides of the paper.
- You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.
- Attempt all questions and write your answers in the space provided.
- Marks are indicated next to each question; the total number of marks is 25.
- You may use a McMaster standard Casio fx-991 MS or MS Plus calculator (no communication capability); no other aids are not permitted.
- Use pen to write your test. If you use a pencil, your test will not be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	Total
Points	5	5	5	5	5	25
Score						

continued ...

1. (5 marks) Put your answer in the space provided for each part.

(a) If an $n \times n$ complex matrix A satisfies $Ax \cdot Ay = x \cdot y$ for all $x, y \in C^n$ then A is unitary. True or False.

T

(b) Compute the inverse of

$$\begin{pmatrix} 0 & 1 & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \end{pmatrix}.$$

$$\begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}$$

(c) If A is similar to B then B is similar to A. True or False.

T

(d) The eigenvalues of a normal matrix are real. True or False.

F

(e) The matrix

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

is unitary. True or False.

T

continued . . .

2. (5 marks) Suppose that V is the subspace of differentiable functions on the real numbers generated by $\{1, x, e^{-x}, xe^{-x}\}$. Consider the linear operator D on V defined by $D(f) = f'$, the derivative of f. Display the matrix for D relative to the basis $B = \{1, x, e^{-x}, xe^{-x}\}$.

$D(1) = 0$, $D(x) = 1$, $D(e^{-x}) = -e^{-x}$, $D(xe^{-x}) = e^{-x} - xe^{-x}$

So $[D]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

3. Suppose that $T\colon P_2 \to P_2$ is the linear transformation given by

$$T(p(x)) = p(x+1)$$

where P_2 is the vector space of polynomials with complex coefficients of degree at most 2.

(a) (3 marks) Compute the determinant of T.

If $B = \{1, x, x^2\}$ then $T(1) = 1$, $T(x) = x+1$, $T(x^2) = (x+1)^2 = x^2+2x+1$

so $[T]_B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ and $\det(T) = \det([T]_B) = 1$

(b) (2 marks) Determine the eigenvalues and eigenvectors for T.

By observation from a), the only eigenvalue is 1.
The eigenvectors for 1 have B coordinates satisfying.

$\begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$ so $b = c = 0$ and the

eigenvectors are the constant polys $p(x) = a$ for $a \in \mathbb{C}$.

continued ...

4. Suppose that A is the Hermitian matrix

$$A = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{i}{2} \\ 0 & -1 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

which has eigenvalues $1, 0$ and -1 and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(a) (3 marks) Determine a matrix P which unitarily diagonalizes A.

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad \text{so} \quad P^*AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = D.$$

(b) (2 marks) Compute A^{100}.

$$A = PDP^* \quad \text{so} \quad A^{100} = PD^{100}P^* \quad \text{and} \quad D^{100} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$PD^{100}P^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 & -\frac{i}{2} \\ 0 & 1 & 0 \\ \frac{i}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

continued . . .

5. (a) (2 marks) For a complex matrix A, define what it means for A to be normal.

If A is an $n \times n$ complex matrix then A is normal if $AA^* = A^*A$.

(b) (3 marks) Prove that if an n x n complex matrix A is unitarily diagonalizable and has real eigenvalues then A is Hermitian.

Let U be an unitary matrix s.t. $U^*AU = D$ where D is diagonal. Moreover, since A has real eigenvalues then D is a real matrix. Then $A = UDU^*$.

To see that A is Hermitian, show $A^* = A$.

$$A^* = (UDU^*)^* = (U^*)^* D^* U^*$$

$$= UDU^* \quad \text{since } D \text{ is real.}$$

$$= A.$$

So A is Hermitian.

THE END