1. (5 points) Determine if the following statements are true or false.

(a) If V is an n-dimensional vector space and $T : V \to V$ is a one-to-one linear operator then the range of T is V.

TRUE

(b) If A and B are similar then they have the same characteristic polynomial.

TRUE

(c) The real vector spaces of 3×3 real matrices and polynomials of degree at most 6 with real coefficients are isomorphic.

The vector space $M_{3,3}(\mathbb{R})$ is 9-dimensional
P_6 is 7-dimensional

FALSE

(d) The eigenvalues of an Hermitian matrix are real.

TRUE

(e) If T is an invertible linear transformation then its kernel is the zero subspace.

TRUE

continued ...

Note!
this ~~is~~ question
is the Pythagorean
theorem

2. (5 points) Place your answer on the line provided.

(a) In an inner product space, if u and v are orthogonal then compute $\|2u - v\|$.

$$\|2u-v\|^2 = \langle 2u-v, 2u-v\rangle$$
$$= \langle 2u,2u\rangle - \langle 2u,v\rangle - \langle v,2u\rangle + \langle v,v\rangle$$
$$= \langle 2u,2u\rangle + \langle v,v\rangle \quad \text{since } u \text{ and } v \text{ orthogonal}$$
$$= 4\|u\|^2 + \|v\|^2$$

$\sqrt{4\|u\|^2 + \|v\|^2}$

(b) Compute the length of $(-2, i)$ in C^2 with the usual Euclidean norm.

$$\left\|\begin{bmatrix}-2\\ i\end{bmatrix}\right\|^2 = \begin{bmatrix}-2 & -i\end{bmatrix}\begin{bmatrix}-2\\ i\end{bmatrix} = 4+1$$

$\sqrt{5}$

So $\left\|\begin{bmatrix}-2\\ i\end{bmatrix}\right\| = \sqrt{5}$

(c) A matrix cannot be similar to itself; true or false.

square

A matrix is always similar to itself since

$$A = I_n A I_n^{-1}$$

FALSE

(d) Compute

$$\int_0^{2\pi} \cos(4x)\cos(3x)\,dx$$

Note that this is $\langle \cos(4x), \cos(3x)\rangle$

w.r.t. to inner product $\int_0^{2\pi} f(x)g(x)\,dx$.

0

As shown in class, $\cos(x), \cos(2x), \ldots$ are orthogonal w.r.t. to this inner product.

So, $\langle \cos(4x), \cos(3x)\rangle = 0$

(e) Name the conic section defined by

$$3x^2 + 4y^2 = 9.$$

Ellipse

continued ...

3. (5 points) Let R^4 have the standard Euclidean inner product. Use the Gram-Schmidt process to find an orthonormal basis for the subspace spanned by

$$u_1 = (1,2,2,0),\ u_2 = (1,3,1,1) \text{ and } u_3 = (1,4,0,1)$$

$\cancel{\text{Let}}$ $v_1 = u_1$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1 = \begin{bmatrix}1\\3\\1\\1\end{bmatrix} - \frac{1+6+2}{1+4+4}\begin{bmatrix}1\\2\\2\\0\end{bmatrix} = \begin{bmatrix}1\\3\\1\\1\end{bmatrix} - \begin{bmatrix}1\\2\\2\\0\end{bmatrix} = \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2 = \begin{bmatrix}1\\4\\0\\1\end{bmatrix} - \frac{1+8}{9}\begin{bmatrix}1\\2\\2\\0\end{bmatrix} - \frac{4+1}{1+1+1}\begin{bmatrix}0\\1\\-1\\1\end{bmatrix}$$

$$= \begin{bmatrix}1\\4\\0\\1\end{bmatrix} - \begin{bmatrix}1\\2\\2\\0\end{bmatrix} - \begin{bmatrix}0\\5/3\\-5/3\\5/3\end{bmatrix} = \begin{bmatrix}0\\1/3\\-1/3\\-2/3\end{bmatrix}$$

So $\left\{\begin{bmatrix}1\\2\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\1/3\\-1/3\\-2/3\end{bmatrix}\right\}$ is an orthogonal basis

We now need to normalize

$$v_1 = \begin{bmatrix}1\\2\\2\\0\end{bmatrix} \Rightarrow \vec{v}_1' = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = \begin{bmatrix}1/3\\2/3\\2/3\\0\end{bmatrix} \qquad \vec{v}_2' = \frac{1}{\sqrt{3}}\begin{bmatrix}0\\1\\-1\\1\end{bmatrix} = \begin{bmatrix}0\\1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix} \qquad \vec{v}_3' = \frac{1}{\sqrt{6/9}}\begin{bmatrix}0\\1/3\\-1/3\\-2/3\end{bmatrix} = \begin{bmatrix}0\\1/\sqrt{6}\\-1/\sqrt{6}\\-2/\sqrt{6}\end{bmatrix}$$

So $\left\{\begin{bmatrix}1/3\\2/3\\2/3\\0\end{bmatrix}, \begin{bmatrix}0\\1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}, \begin{bmatrix}0\\1/\sqrt{6}\\-1/\sqrt{6}\\-2/\sqrt{6}\end{bmatrix}\right\}$ is an orthonormal basis

continued . . .

4. (5 points)

(a) Suppose that $B = \{f_1, f_2, f_3\}$ is a basis for a subspace V of real-valued functions defined on the real line where

$$f_1 = e^{-x},\ f_2 = xe^{-x} \text{ and } f_3 = x^2e^{-x}.$$

Let $D : V \to V$ be the linear operator differentiation with respect to x. Find the matrix for D with respect to the basis B.

$D(f_1) = -e^{-x}$ $\quad$ $D(f_2) = -xe^{-x} + e^{-x}$ $\quad$ $D(f_3) = -x^2e^{-x} + 2xe^{-x}$

$[D(f_1)]_B = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ $\quad$ $[D(f_2)]_B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\quad$ $[D(f_3)]_B = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

So $[D]_{B,B} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

(b) Use the matrix from part (a) to compute $D(3e^{-x} + xe^{-x} - 2x^2e^{-x})$.

Note that $[3e^{-x} + xe^{-x} - 2x^2e^{-x}]_B = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

So $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 2 \end{bmatrix}_B \Rightarrow D(3e^{-x} + xe^{-x} - 2x^2e^{-x})$

$= -2e^{-x} - 5xe^{-x} + 2x^2e^{-x}$

these are the coefficients of in basis B, i.e.

$[D(3e^{-x} + xe^{-x} - 2x^2e^{-x})]_B$

continued . . .

5. (5 points)Find a unitary matrix P which unitarily diagonalizes A and determine $P^{-1}AP$ where

$$A = \begin{pmatrix} 3 & -i \\ i & 3 \end{pmatrix}.$$

$$\lambda I_2 - A = \begin{bmatrix} \lambda-3 & i \\ -i & \lambda-3 \end{bmatrix} \Rightarrow \det(\lambda I_2 - A) = (\lambda-3)^2 + i^2$$
$$= \lambda^2 - 6\lambda + 9 - 1$$
$$= \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2)$$

To find P, we need to find unit eigenvectors for each eigenvalue

$\boxed{\lambda = 2}$ $\begin{bmatrix} 2-3 & i \\ -i & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & i \\ -i & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$ eigenvector $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

Normalized $\vec{u}_1 = \frac{1}{\|\vec{v}\|}\begin{bmatrix} i \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\boxed{\lambda = 4}$

$\begin{bmatrix} 4-3 & i \\ -i & 4-3 \end{bmatrix} = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$ eigenvector $\vec{w} = \begin{bmatrix} i \\ -1 \end{bmatrix}$

normalized $\vec{u}_2 = \frac{1}{\|\vec{w}\|}\begin{bmatrix} i \\ -1 \end{bmatrix} = \begin{bmatrix} i/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$$\boxed{\text{So } P = \begin{bmatrix} i/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}}$$

continued . . .

6. (a) (2 points) In the inner product space of continuous functions on $[-1,1]$ with inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} fg\,dx,$$

compute the inner product of 1 with x^2.

$\langle 1, x^2\rangle = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \left(\frac{-1}{3}\right) = \boxed{\frac{2}{3}}$

(b) (3 points) Suppose that A is an $n \times n$ invertible matrix. Show that for $u, v \in R^n$,

$$(u^T A^T A v)^2 \leq (u^T A^T A u)(v^T A^T A v).$$

This is Cauchy-Schwarz!

Use standard inner product on $\mathbb{R}^n$. Then Cauchy Schwarz implies $\langle A\vec{u}, A\vec{v}\rangle^2 \leq \|A\vec{u}\|^2 \|A\vec{v}\|^2$

But $\langle A\vec{u}, A\vec{v}\rangle = (A\vec{u})^T A\vec{v} = \vec{u}^T A^T A\vec{v}$

$\langle A\vec{u}, A\vec{u}\rangle = u^T A^T A\vec{u}$ and $\langle A\vec{v}, A\vec{v}\rangle = \vec{v}^T A^T A\vec{v}$.

So $(\vec{u}^T A^T A\vec{v})^2 = \langle A\vec{u}, A\vec{v}\rangle^2 \leq \|A\vec{u}\|^2 \|A\vec{v}\|^2$

$= (u^T A^T A\vec{u})(\vec{v}^T A^T A\vec{v})$

continued . . .

7. (5 points) Diagonalize the quadratic form

$$x^2 - 2y^2 - 2z^2 + 4yz$$

and say if it is positive definite, negative definite or indefinite.

$$Q(\vec{x}) = [x\ y\ z]\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Find eigenvalues $\lambda I_3 - A = \begin{bmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda+2 & -2 \\ 0 & -2 & \lambda+2 \end{bmatrix}$

$$\Rightarrow \det(\lambda I_3 - A) = (\lambda-1)(\lambda+2)(\lambda+2) - 4(\lambda-1) = (\lambda-1)(\lambda^2+4\lambda)$$
$$= (\lambda-1)(\lambda^2+4\lambda+4-4) = \lambda(\lambda-1)(\lambda+4)$$

Since A is orthogonally diagonalizable, there exists an orthogonal matrix P such that $A = P\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}P^T$. Let $\vec{x} = P\vec{y}$. Then

$$Q_A(\vec{x}) = Q_D(\vec{y}) = 0y_1^2 + 1\cdot y_2^2 + (-4)y_3^2$$
$$\boxed{= y_2^2 - 4y_3^2}$$

Since we have both positive, negative & zero eigenvalues, quadratic form is $\boxed{\text{indefinite}}$

continued . . .

8. (5 points) Let A be a normal matrix. Prove that for all $x \in \mathbf{C}^n$ that $\|Ax\| = \|A^*x\|$.

$\|A\vec{x}\|^2 = \langle A\vec{x}, A\vec{x} \rangle$

$= (A\vec{x})^* A\vec{x}$ (def^n of standard inner product on $\mathbb{C}^n$

$= (\vec{x}^* A^*)(A\vec{x})$ (properties of conjugate transpose)

$= \vec{x}^* (A^* A)\vec{x}$

$= \vec{x}^* A A^* \vec{x}$ since A is normal, $AA^* = A^*A$

$= \vec{x}^* A^{**} A^* \vec{x}$ since $A = A^{**}$

$= (A^*\vec{x})^* A^* \vec{x}$

$= \langle A^*\vec{x}, A^*\vec{x} \rangle = \| A^* \vec{x} \|^2$.

So $\|A\vec{x}\|^2 = \|A^*\vec{x}\|^2$.
Because both $\|A\vec{x}\|$ and $\|A^*\vec{x}\|$ are positive, we have $\|A\vec{x}\| = \|A^*\vec{x}\|$.

THE END