Orthogonal complement

Definition

If W is a subspace of an inner product space V then we say that $v \in V$ is orthogonal to W if v is orthogonal to every $w \in W$. The set of all $v \in V$ which are orthogonal to W is called the orthogonal complement of W and is written $W^\perp$.

Theorem

If W is a subspace of an inner product space V then

1. *$W^\perp$ is a subspace of V,*
2. *W and $W^\perp$ have only 0 in their intersection, and*
3. *if V is finite-dimensional then $(W^\perp)^\perp = W$.*

Connection with matrices

Theorem

Suppose that A is any $m \times n$ matrix. Then

1. *the nullspace of A and the row space of A are orthogonal complements in R^n with respect to the usual (Euclidean) inner product on R^n.*
2. *the nullspace of A^T and the column space of A are orthogonal complements in R^m with respect to the Euclidean inner product on R^m.*

Orthogonal sets

Definition

A set S of non-zero vectors in an inner product space is called

- orthogonal if every distinct pair of vectors in S is orthogonal.
- It is called orthonormal if it is orthogonal and every vector has length one.
- It is called an orthonormal (or orthogonal) basis if it is orthonormal (or orthogonal) and a basis.

Theorem (6.3.1)

If S is an orthogonal set of non-zero vectors in an inner product space then S is linearly independent.

Properties of orthonormal bases

Theorem (6.3.2)

(a) If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal basis for an inner product space V then for every $u \in V$,

$$u = \frac{\langle u, v_1\rangle}{\|v_1\|^2}v_1 + \frac{\langle u, v_2\rangle}{\|v_2\|^2}v_2 + \ldots + \frac{\langle u, v_n\rangle}{\|v_n\|^2}v_n$$

(b)If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for an inner product space V then for every $u \in V$,

$$u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \ldots + \langle u, v_n\rangle v_n$$

Properties of orthonormal bases, cont'd

Theorem

If S is an orthonormal basis for an n-dimensional inner product space V and then for every $u, v \in V$, if

$$u = (u_1, u_2, \ldots, u_n)_S \text{ and } v = (v_1, v_2, \ldots, v_n)_S$$

then

1. $||u|| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$
2. $d(u, v) = \sqrt{(u_1 - v_1)^2 + \ldots (u_n - v_n)^2}$
3. $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$

Projections

Theorem (6.3.3)

If W is a finite-dimensional subspace of an inner product space V then every vector $u \in V$ can be written as

$$u = w_1 + w_2$$

where $w_1 \in W$ and $w_2 \in W^\perp$. In fact, this representation of u is unique.

Notation

In the previous theorem, w_1 is called the orthogonal projection of u on W and is written $proj_W(u)$.

Finding orthonormal bases

Theorem

If $\{v_1, v_2, \ldots, v_r\}$ is an orthonormal basis for a subspace W of an inner product space V then for any $u \in V$,

$$proj_W(u) = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \ldots + \langle u, v_r\rangle v_r$$

Theorem (6.3.5)

Every non-zero finite-dimensional inner product space has an orthonormal basis.