Orthogonal matrices

Definition

A square matrix A is called orthogonal if $A^{-1} = A^T$.

Theorem

The following are equivalent for an $n \times n$ matrix A:

1. *A is orthogonal.*
2. *The rows of A form an orthonormal basis for R^n.*
3. *The columns of A form an orthonormal basis for R^n.*

Orthogonal matrices, cont'd

Theorem

- *The inverse of an orthogonal matrix is orthogonal.*
- *A product of orthogonal matrices is orthogonal.*
- *The determinant of an orthogonal matrix is* ± 1.

Theorem

If A *is* $n \times n$ *then the following are equivalent:*

1. A *is orthogonal.*
2. $||Ax|| = ||x||$ *for all* $x \in R^n$.
3. $Ax \cdot Ay = x \cdot y$ *for all* $x, y \in R^n$.

Theorem

If P *is a transition matrix from one orthonormal basis to another then* P *is orthogonal.*

Orthogonal diagonalization

Definition

If P diagonalizes A and P is orthogonal then A is said to be orthogonally diagonalizable. That is, there is an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

Theorem

If A is an $n \times n$ real matrix then the following are equivalent:

1. *A is orthogonally diagonalizable.*
2. *A has an orthonormal set of n eigenvectors.*
3. *A is symmetric.*

Symmetric matrices

Theorem

If A is a symmetric matrix then

1. *The eigenvalues of A are all real numbers.*
2. *Eigenvectors from different eigenspaces are orthogonal.*

Definition

1. An $n \times n$ complex matrix A is Hermitian if $A^* = A$; remember that $A^* = \overline{A^T}$, the conjugate of the transpose.
2. An $n \times n$ complex matrix U is called unitary if $A^* = A^{-1}$.

Theorem

If A is a Hermitian matrix then

1. *The eigenvalues of A are all real numbers.*
2. *Eigenvectors from different eigenspaces are orthogonal.*

Properties of unitary matrices

Theorem

The following are equivalent for an $n \times n$ complex matrix A:

1. *A is unitary.*
2. *The rows of A form an orthonormal basis for C^n.*
3. *The columns of A form an orthonormal basis for C^n.*

Theorem

- *The inverse of an unitary matrix is unitary.*
- *A product of unitary matrices is unitary.*
- *The determinant of an unitary matrix is of norm 1.*

Properties of unitary matrices, cont'd

Theorem

If A is $n \times n$ then the following are equivalent:

1. *A is unitary.*
2. $||Ax|| = ||x||$ *for all* $x \in C^n$.
3. $Ax \cdot Ay = x \cdot y$ *for all* $x, y \in C^n$.

Theorem

If P is a transition matrix from one orthonormal basis to another in a complex space then P is unitary.

Unitary diagonalization

Definition

Suppose A is an $n \times n$ complex matrix. Then if P diagonalizes A and P is unitary then A is said to be unitarily diagonalizable. That is, there is an unitary matrix P such that $P^{-1}AP$ is diagonal.

Theorem

If A is an $n \times n$ complex matrix then the following are equivalent:

1. *A is unitarily diagonalizable and has real eigenvalues.*
2. *A has real eigenvalues and an orthonormal set of n eigenvectors.*
3. *A is Hermitian.*

Normal matrices and Schur's Theorem

Definition

A complex $n \times n$ matrix A is called normal if $A^*A = AA^*$.

Theorem

If A is an $n \times n$ complex matrix then the following are equivalent:

1. *A is unitarily diagonalizable.*
2. *A has an orthonormal set of n eigenvectors.*
3. *A is normal.*

Theorem (Schur's theorem)

If A is any $n \times n$ complex matrix then there is an upper triangular matrix S and a unitary matrix P such that $A = P^{-1}SP$.