Matrices with complex entries

- From now on, unless it is explicitly said otherwise, matrices will be assumed to have complex entries.
- All basic linear algebra - linear equations with complex coefficients, matrix multiplication and addition, determinant calculations - work exactly the same over the complex numbers as they do over the reals.
- In particular, a square matrix is invertible iff its determinant is non-zero.
- The biggest advantage of using the complex numbers is that characteristic polynomials will always have roots so every square complex matrix has at least one eigenvalue.
Vector Space Axioms

Suppose $V$ is a set together with the operations $+$ and multiplication by scalars (real numbers). Then we call $V$ a (real) vector space if the following axioms are satisfied:

1. If $u$ and $v$ are objects in $V$, then $u + v$ is in $V$;
2. For all $u$ and $v$ in $V$, $u + v = v + u$;
3. For all $u$, $v$ and $w$ in $V$, $u + (v + w) = (u + v) + w$;
4. There is an object $0$ in $V$ such that for all $u$ in $V$, $0 + u = u$;
5. For all $u$ in $V$, there is an object $-u$ in $V$ such that $u + (-u) = 0$;
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5. For all $u$ in $V$, there is an object $-u$ in $V$ such that $u + (-u) = 0$;
6. For any scalar $k$ and any $u$ in $V$, $ku$ is in $V$;
7. For any scalar $k$ and $u$, $v$ in $V$, $k(u + v) = ku + kv$;
8. For scalars $k$ and $m$, and any $u$ in $V$, $(k + m)u = ku + mu$;
9. For scalars $k$ and $m$, and any $u$ in $V$, $k(mu) = (km)u$; and
10. For all $u$ in $V$, $1u = u$. 
Definition

A subset \( W \) of a vector space \( V \) is a subspace of \( V \) if \( W \) is a vector space under the addition and scalar multiplication defined on \( V \).
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**Theorem**

A subset $W$ of a vector space $V$ is a subspace of $V$ if

1. $W$ is closed under $+$ i.e. if $u$ and $v$ are in $W$ then $u + v$ is in $W$, and
2. $W$ is closed under scalar multiplication i.e. if $k$ is a scalar and $u$ is in $W$ then $ku$ is in $W$. 
Definition

If $S = \{v_1, v_2, \ldots, v_r\}$ is a non-empty set of vectors such that the only solution for scalars $k_1, k_2, \ldots, k_r$ of the equation

$$k_1 v_1 + k_2 v_2 + \ldots + k_r v_r = 0$$

is $k_1 = k_2 = \ldots = k_r = 0$ then $S$ is said to be linearly independent. Otherwise, $S$ is linearly dependent.
Definition

If $V$ is a vector space and $S = \{v_1, v_2, \ldots, v_n\}$ is a set of vectors in $V$ then $S$ is said to be a basis for $V$ if

1. $S$ is linearly independent and
2. $S$ spans $V$. 

Definition

A vector space $V$ is called finite-dimensional if it has a finite basis. Otherwise it is called infinite-dimensional.

Theorem (4.5.1)

If $V$ is a finite-dimensional vector space then all bases for $V$ have the same number of vectors.
Basis and Dimension

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*If $V$ is a finite-dimensional vector space then all bases for $V$ have the same number of vectors.*