Vector Space Axioms

Suppose V is a set together with the operations $+$ and multiplication by scalars (real numbers). Then we call V a (real) vector space if the following axioms are satisfied:

1. If u and v are objects in V, then $u+v$ is in V;
2. For all u and v in V, $u+v=v+u$;
3. For all u, v and w in V, $u+(v+w)=(u+v)+w$;
4. There is an object 0 in V such that for all u in V, $0+u=u$;
5. For all u in V, there is an object $-u$ in V such that $u+(-u)=0$;
6. For any scalar k and any u in V, ku is in V;
7. For any scalar k and u, v in V, $k(u+v)=ku+kv$;
8. For scalars k and m, and any u in V, $(k+m)u=ku+mu$;
9. For scalars k and m, and any u in V, $k(mu)=(km)u$; and
10. For all u in V, $1u=u$.

Subspaces

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Theorem

A non-empty subset W of a vector space V is a subspace of V if

1. *W is closed under $+$ i.e. if u and v are in W then $u + v$ is in W, and*
2. *W is closed under scalar multiplication i.e. if k is a scalar and u is in W then ku is in W.*

Linear independence

Definition

If $S = \{v_1, v_2, \ldots, v_r\}$ is a non-empty set of vectors such that the only solution for scalars $k_1, k_2, \ldots, k_r$ of the equation

$$k_1 v_1 + k_2 v_2 + \ldots + k_r v_r = 0$$

is $k_1 = k_2 = \ldots = k_r = 0$ then S is said to be linearly independent. Otherwise, S is linearly dependent.

Definition

If V is a vector space and $S = \{v_1, v_2 \ldots, v_n\}$ is a set of vectors in V then S is said to be a basis for V if

1. S is linearly independent and
2. S spans V.

Basis and Dimension

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Definition

A vector space V is called finite-dimensional if it has a finite basis. Otherwise it is called infinite-dimensional.

Theorem (4.5.1)

If V is a finite-dimensional vector space then all bases for V have the same number of vectors.

Complex vector spaces

- Suppose V is a set together with the operations $+$ and multiplication by complex numbers i.e. the scalars are now complex. Then we call V a complex vector space if the same 10 axioms from section 4.1 are satisfied.
- The definition of subspace remains the same for complex vector spaces; the main Theorem for identifying subspaces is also the same i.e. it is sufficient for a subset of a vector space to be closed under $+$ and scalar multiplication to be a subspace.
- Some things do change:

Definition

If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are vectors in C^n then we define the dot product as

$$u \cdot v = u_1\bar{v}_1 + u_2\bar{v}_2 + \ldots + u_n\bar{v}_n$$

Properties of the dot product

Theorem (5.3.1)

If u, v and w are vectors in C^n and k is any complex number (scalar) then

1. $u \cdot v = \overline{v \cdot u}$,
2. $(u + v) \cdot w = u \cdot w + v \cdot w$,
3. $(ku) \cdot v = k(u \cdot v)$, *and*
4. $u \cdot u \geq 0$. *Moreover* $u \cdot u = 0$ *iff* $u = 0$.

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The complex norm

For $u = (u_1, u_2, \ldots, u_n)$ in C^n, we define

$$||u|| = \sqrt{u \cdot u} = \sqrt{|u_1|^2 + |u_2|^2 + \ldots + |u_n|^2}$$