Complex vector spaces

- Suppose V is a set together with the operations $+$ and multiplication by complex numbers i.e. the scalars are now complex. Then we call V a complex vector space if the same 10 axioms from section 4.1 are satisfied.
- The definition of subspace remains the same for complex vector spaces; the main Theorem for identifying subspaces is also the same i.e. it is sufficient for a subset of a vector space to be closed under $+$ and scalar multiplication to be a subspace.
- Some things do change:

Definition

If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are vectors in C^n then we define the dot product as

$$u \cdot v = u_1 \bar{v}_1 + u_2 \bar{v}_2 + \ldots + u_n \bar{v}_n$$

Properties of the dot product

Theorem (5.3.1)

If u, v and w are vectors in C^n and k is any complex number (scalar) then

1. $u \cdot v = \overline{v \cdot u}$,
2. $(u + v) \cdot w = u \cdot w + v \cdot w$,
3. $(ku) \cdot v = k(u \cdot v)$, *and*
4. $u \cdot u \geq 0$. *Moreover* $u \cdot u = 0$ *iff* $u = 0$.

The complex norm

For $u = (u_1, u_2, \ldots, u_n)$ in C^n, we define

$$||u|| = \sqrt{u \cdot u} = \sqrt{|u_1|^2 + |u_2|^2 + \ldots + |u_n|^2}$$

Linear independence and bases in complex vector spaces

- Linear independence in complex vector spaces is identical to linear independence in real vector spaces with the only change being that the scalars are complex.
- A basis for a complex vector space is a maximal linearly independent subset of that space.
- Every complex vector space has a basis and the size of the basis is determined by the space itself so in particular if the space is finite-dimensional then all bases have the same size.

How to form a basis

Theorem (Plus/Minus Theorem, 4.5.3)

Let S be a non-empty subset of a vector space V.

1. *If S is linearly independent and v is in V but not in the span of S then $S \cup \{v\}$ is linearly independent.*
2. *If v in S is expressible as a linear combination of other vectors from S then the spans of S and $S \setminus \{v\}$ (S without v) are the same.*

Coordinates relative to a basis

Theorem (4.4.1)

If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V then every v in V can be written as

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

for a unique choice of scalars $c_1, c_2, \ldots, c_n$.

Eigenvalues

Definition

Suppose that A is an $n \times n$ complex matrix, λ is a scalar and $x \in \mathbb{C}^n$ is non-zero such that

$$Ax = \lambda x$$

Then λ is called an eigenvalue of A and x is called an eigenvector.

Eigenvalues, cont'd

Theorem

If A is an $n \times n$ matrix and λ is a scalar then the following are equivalent:

1. *λ is an eigenvalue of A.*
2. *The system of linear equations $(\lambda I - A)x = 0$ has non-trivial solutions.*
3. *There is a non-zero $x \in \mathbb{C}^n$ such that $Ax = \lambda x$.*
4. *λ is a solution to the characteristic equation $\det(\lambda I - A) = 0$.*

Definition

If λ is an eigenvalue for A, an $n \times n$ matrix, then the set of all x such that $Ax = \lambda x$ forms a subspace of $\mathbb{C}^n$ which is called the eigenspace of A corresponding to λ.

Diagonalizability

Definition

A square matrix A is called diagonalizable if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. P is said to diagonalize A.

Theorem (5.2.1)

The following are equivalent for an $n \times n$ matrix A:

1. *A is diagonalizable.*
2. *A has n linearly independent eignevectors.*