1. (5 marks) Put your answer in the space provided for each part.

   (a) For all complex numbers z, $|z| = |\bar{z}|$. True or false.

   T

   (b) $1, 1+x$ and $x^2 + x^3$ is a basis for the vector space of polynomials of degree less than or equal to 3. True or false.

   F

   (c) If $u = (3, 1+2i, 2)$ and $v = (i, 1-i, 0)$ in C^3 then $u \cdot v =$ -1

   $3(-i) + (1+2i)(1+i) = -3i + 1 - 2 + 3i$

   (d) Suppose V is a real inner product space and $u, v \in V$ such that $\langle u, u\rangle = 2$, $\langle u, v\rangle = -1$ and $\langle v, v\rangle = 1$. Compute $||u+v||$.

   $\sqrt{\langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle} = \sqrt{2-2+1} = 1$

   1

   (e) In the inner product space of continuous functions on $[-1, 1]$ with inner product given by

   $$\langle f, g\rangle = \int_{-1}^{1} fg dx,$$

   compute the inner product of 1 with x^2.

   $2/3$ $\quad \int_{-1}^{1} x^2 dx = \frac{2}{3}$

continued ...

2. (a) (2 marks) Express $\frac{i}{1+i}$ in the form $a + bi$.

$$\frac{i}{i+1} \cdot \frac{1-i}{1-i} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2} i$$

(b) (3 marks) Find all complex numbers z such that $z^4 = -1$.

Let $z = re^{i\theta} = 1 \cdot e^{i\pi} = -1$

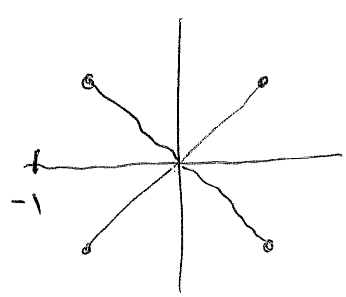
Then $z^{1/4} = 1 \cdot \left(e^{i\pi + 2k\pi}\right)^{1/4}$ for $k \in \mathbb{N}$

$$= 1 \cdot e^{i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)}$$

The values for $k = 0, 1, 2$ and 3 are distinct. so

$$z = \cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + i \sin\left(\frac{\cancel{k}\pi}{4} + \frac{k\pi}{2}\right) \text{ for } k = 0, 1, 2, 3$$

$$= \pm \frac{1}{\sqrt{2}} \pm i \frac{1}{\sqrt{2}}$$



continued ...

3. (5 marks) Show that the set of continuous real-valued functions f on $[0, 1]$ which satisfy $f(0) = f(1) = 0$ is a subspace of all continuous functions on $[0, 1]$.

The ~~vector space~~ set of continuous functions on $[0,1]$ is a vector space so it suffices to show that our set W is closed under + and scalar multiplication.

+: Suppose f, g are continuous on $[0,1]$ and

$f(0) = f(1) = g(0) = g(1) = 0$.

Then $(f+g)(0) = f(0) + g(0) = 0$

$(f+g)(1) = f(1) + g(1) = 0$

so $f+g \in W$.

Scalars: If k is a scalar and $f \in W$ then

$f(0) = f(1) = 0$ and $(kf)(0) = k(f(0)) = 0$

$(kf)(1) = k(f(1)) = 0$

So $kf \in W$.

So W is a subspace.

4. (5 marks) Suppose A is the invertible matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

We know that the function given by $\langle u, v\rangle = Au \cdot Av$ is an inner product on R^3. Compute the distance between $(1,0,0)$ and $(0,1,0)$ with respect to this inner product.

If $u = (1,0,0)$, $v = (0,1,0)$ then $d(u,v) = \|u-v\|$

$= \sqrt{\langle u-v, u-v\rangle}$. $u - v = (1,-1,0)$

and $A(u-v) = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ so

$d(u,v) = \sqrt{(2,-1,1)\cdot(2,-1,1)}$

$= \sqrt{6}$

continued ...

#5 Produce an orthogonal basis using G-S starting with $\{1, x, x^2\}$.

① $1 \neq 0$ and $\|1\|^2 = 1+1+1 = 3$ so $\|1\| = \sqrt{3}$.

② $\langle x, 1\rangle = -1+0+1 = 0$ so x is already orthogonal to 1. $\|x\|^2 = 1+0+1 = 2$ so $\|x\| = \sqrt{2}$.

③ $\langle x^2, 1\rangle = 1+0+1 = 2$ and.
$\langle x^2, x\rangle = -1+0+1 = 0$.

If W is a subspace generated by 1 and x then

$$\text{proj}_W x^2 = \text{proj}_1 x^2 + \text{proj}_x x^2$$
$$= \frac{\langle x^2, 1\rangle}{\|1\|^2} 1 + \frac{\langle x^2, x\rangle}{\|x\|^2} x$$
$$= \frac{2}{3}$$

So $x^2 - \frac{2}{3}$ is orthogonal to 1 and x and an orthogonal basis is $\{1, x, x^2 - \frac{2}{3}\}$.