Mathematics 2R3 Test 1

Dr. Bradd Hart Oct. 9, 2019

Last Name: Solutions Initials:

Student No.:

- The test is 50 minutes long.
- The test has 6 pages and 5 questions and is printed on BOTH sides of the paper.
- You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.
- Attempt all questions and write your answers in the space provided.
- Marks are indicated next to each question; the total number of marks is 25.
- You may use a McMaster standard Casio fx-991 MS or MS Plus calculator (no communication capability); no other aids are not permitted.
- Use pen to write your test. If you use a pencil, your test will not be accepted for regrading (if needed).

Good Luck!

Score

Question	1	2	3	4	5	Total
Points	5	5	5	5	5	25
Score						

continued ...

1. (5 marks) Put your answer in the space provided for each part.

(a) Every complex $n \times n$ matrix has an eigenvalue. True or false.

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(b) Every linearly independent set of n vectors in $\mathbf{R}^n$ is a basis for $\mathbf{R}^n$. True or false.

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(c) There are 4 polynomials in the vector space P_5 of real polynomials of degree less than or equal to 5 that span P_5. True or false.

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(d) If $u = (i, -i)$ and $v = (-i, -i)$ in $\mathbf{C}^2$ then $u \cdot v =$ 0

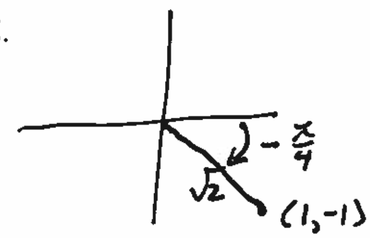
(e) Find the eigenvalues of the following matrix by inspection.

$$\begin{pmatrix} 3-i & 0 & 0 \\ -1 & i & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

$3-i, i, 1$

continued ...

2. (a) (3 marks) Find all complex numbers z such that $z^4 = 1 - i$.



$1 - i = \sqrt{2}\, e^{-i\frac{\pi}{4}}$

So one 4^{th} root is $2^{1/8} e^{-i\frac{\pi}{16}}$.

All 4^{th} roots are obtained by multiplying by the 4^{th} roots of unity $e^{i\frac{k\pi}{2}}$ for $k = 0, 1, 2, 3$.

So all 4^{th} roots are $\sqrt{2}\, e^{-i\frac{\pi}{16} + i\frac{k\pi}{2}}$ for $k = 0, 1, 2, 3$

(b) (2 marks) Suppose V is a complex inner product space and $u, v \in V$ such that $\langle u, u\rangle = 1$, $\langle u, v\rangle = i$ and $\langle v, v\rangle = 1$. Compute $\|u + v\|$.

$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$

$= 1 + i - i + 1 = 2$

so $\|u+v\| = \sqrt{2}$

continued . . .

3. (5 marks) Suppose A is the matrix

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors for A in $\mathbf{C}^2$.

Char poly is $\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -i \\ i & \lambda-1 \end{vmatrix} = (\lambda-1)^2 - 1$
$= (\lambda-2)\lambda$

so the eigenvalues are $\lambda = 0, 2$.

Eigenvectors:

$\lambda = 0$: Solve $\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$ so $ix = y$ and

all eigenvectors are of the form $t\begin{pmatrix} 1 \\ i \end{pmatrix}$ for $t \in \mathbb{C}$.

$\lambda = 2$: Solve $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ so $x = iy$ and

all eigenvectors are of the form $t\begin{pmatrix} i \\ 1 \end{pmatrix}$ for $t \in \mathbb{C}$.

continued ...

4. In the inner product space of continuous functions on $[-1, 1]$ with inner product given by

$$\langle f, g\rangle = \int_{-1}^{1} f g \, dx$$

(a) (3 marks) use the Gram-Schmidt process applied to the linearly independent set $\{1, x, x^2\}$ to form an orthogonal set; you do not have to normalize the vectors.

$v_1 = 1$, $\|v_1\|^2 = \int_{-1}^{1} 1\,dx = 2$, $\langle 1, x\rangle = \int_{-1}^{1} x\,dx = 0$ so

$v_2 = x$.

$\langle x, x^2\rangle = \int_{-1}^{1} x^3 dx = 0$ so $\mathrm{proj}_x x^2 = 0$

$\langle 1, x^2\rangle = \int_{-1}^{1} x^2 dx = \left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{2}{3}$.

So $v_3 = x^2 - \mathrm{proj}_1 x^2 = x^2 - \frac{\left(\frac{2}{3}\right)}{2} = x^2 - \frac{1}{3}$.

The orthogonal basis is then $1, x, x^2 - \frac{1}{3}$.

(b) (2 marks) If f and g are two continuous functions on $[-1, 1]$, explain why

$$\left(\int_{-1}^{1} (f+g)^2 dx\right)^{1/2} \leq \left(\int_{-1}^{1} f^2 dx\right)^{1/2} + \left(\int_{-1}^{1} g^2 dx\right)^{1/2}.$$

This is the triangle inequality applied to the given inner product; in general

$\|f + g\| \leq \|f\| + \|g\|$.

5. Suppose that W is a subspace of an inner product space V.

(a) (2 marks) Define what is meant by the orthogonal complement, $W^{\perp}$, of W.

The orthogonal complement of W, $W^{\perp}$, is the set of all $v \in V$ s.t. $\langle v, w \rangle = 0$ for all $w \in W$.

(b) (3 mark) Prove that $W^{\perp}$ is a subspace of V.

① $W^{\perp}$ is not empty since $0 \in W^{\perp}$.

② If $u, v \in W^{\perp}$ and $w \in W$ then $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
$= 0 + 0 = 0$

so $u + v \in W^{\perp}$.

③ If $u \in W^{\perp}$ and k is a scalar then for any $w \in W$
$\langle ku, w \rangle = k \langle u, w \rangle = k \cdot 0 = 0$.

Since $W^{\perp}$ is non-empty and closed under $+$ and scalar multiplication, $W^{\perp}$ is a subspace of V.

THE END