Math 3EE3, Test 2

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Please write complete answers to all of the questions in the test booklet provided. Partial credit may be given for your work. Unless otherwise noted, you need to justify your solutions in order to receive full credit. Please be sure to include your name and student number on all sheets of paper that you hand in.

1. (5 marks)
   (a) Compute $[Q(\sqrt{3},\sqrt{7}):Q]$.
   (b) Exhibit a basis for $Q(\sqrt{3},\sqrt{7})$ as a Q-vector space.

2. (5 marks) Let $\tau = \sqrt{3}+i$, an element of $\mathbb{C}$, a field extension of Q.
   (a) Argue that τ is algebraic over Q.
   (b) What is the degree of τ over Q?

3. (a) State Zorn's Lemma.
   (b) Use Zorn's Lemma to show that every vector space has a basis.

4. (5 marks) Show that if F is a finite field then the cardinality of F is p^n for some prime p and some natural number $n > 0$.

5. (5 marks) The Hilbert basis theorem tells us that if I is an ideal in $F(x_1,\ldots,x_n)$ then I is finitely generated. Use this to show that if I is an ideal in $F(x_1,\ldots,x_n)$ generated by a set of polynomials S then there is a finite subset $S_0 \subseteq S$ which generates I.