Solutions to Assignment 5

1. a) It is useful for this question to think of a function $f: A \to B$ as a set of ordered pairs $(a, f(a)) \in A \times B$. This way, $f \subseteq A \times B$. Our partial order P is then all partial functions f from K_1 to K_2 s.t. f is the identity on F and $f: dom(f) \xrightarrow{\sim} rng(f)$; the order is given by inclusion.

(P is a set by the way because $P \subseteq \mathcal{P}(K_1 \times K_2)$.)

To see that Zorn's lemma appies, let $\mathcal{C}$ be a chain in P i.e. if $f, g \in \mathcal{C}$ then $f \subseteq g$ or $g \subseteq f$.

Let $f = \bigcup \mathcal{C}$ i.e. the union of all the functions in $\mathcal{C}$.

$dom(f) = \bigcup \{ dom(g) : g \in \mathcal{C} \}$ and $rng(f) = \bigcup \{ rng(g) : g \in \mathcal{C} \}$

If $g \in \mathcal{C}$ then $g \subseteq f$ so as long as $f \in P$, it is an upper bound for $\mathcal{C}$. But to see that $f \in P$ we need to check 1) if $a, b \in dom(f)$ then $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a) f(b)$.

- This is clear by choosing $g \in \mathcal{C}$ s.t. $a, b \in dom(g)$ and then using the fact that g is a homomorphism.

2) f is 1-1 and onto: If $f(a) = 0$ and $a \in dom(g)$ for $g \in \mathcal{C}$ then $g(a) = 0$ and $a = 0$. f is onto ~~if $b \in rng(f)$ then $b \in rng(g)$~~ by definition.

b) So Zorn's Lemma applies and we can choose a maximal f, a partial ~~function~~ isomorphism from K_1 to K_2.

Now if $\mathrm{dom}(f) \neq K_1$ then choose $a \in K_1 \setminus \mathrm{dom}(f)$. a is algebraic / $\mathrm{dom}(f)$. So let p be its minimal poly / $\mathrm{dom}(f)$. p and $f(p)$ are irreducible polys

and $F_1[a] \cong F_1[x]/\langle p\rangle$ where $F_1 = \mathrm{dom}(f)$ and $F_2 = \mathrm{rng}(f)$.

If $F_2 = \mathrm{rng}(f)$ then choose $b \in K_2$ solving $f(p)$; K_2 is alg. closed.

Then $F_2[b] \cong F_2[x]/\langle f(p)\rangle$ and so the map extending f to $a \mapsto b$ is a partial iso. from

$F_1[a]$ to $F_2[b]$. This contradicts the maximality of f.

To see that f is onto K_2, suppose that $b \in K_2 - \mathrm{rng}(f)$. As above, let p be the minimal poly for b over F_2. Then $F_2[b] \cong F_2[x]/\langle p\rangle$ and if we choose

$a \in K_1$ which satisfies $f^{-1}(p)$ then $F_1[a]$ is also iso to $F_2[x]/\langle p\rangle$ extending f. Again this contradicts

the maximality of f. So $f: K_1 \to K_2$ and K_1 and K_2 are isomorphic over F.

2. Suppose that $q(x)$ is an irreducible poly. over $\mathbb{Z}_p$.

Then $F = \mathbb{Z}_p[x]/\langle q \rangle$ is a finite field of size p^n

for some $n \in \mathbb{N}$. But every element of F satisfies $x^{p^n} - x$ and $x/\langle q \rangle$ has $q(x)$ as its minimal poly. So q divides $x^{p^n} - x$.

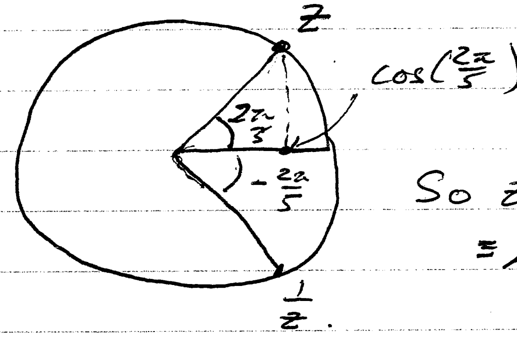
3. Let's see that both $\cos(\frac{2\pi}{5})$ and $\cos(\frac{2\pi}{7})$ are algebraic.

Here is a nice little trick: If $z = e^{\frac{2\pi i}{n}}$ then

$z^n - 1 = 0$ and $z^n - 1 = (z-1)(1 + z + \cdots + z^{n-1})$ and $z \neq 1$

so $1 + z + z^2 + \cdots + z^{n-1} = 0$.

Let's look at $n = 5$:



$1 + z + z^2 + z^3 + z^4 = 0$, $z \neq 0$ so divide by z^2

So $z + \frac{1}{z} = 2\cos(\frac{2\pi}{5})$

$$\frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 = 0. \quad (*)$$

$(z + \frac{1}{z})^2 = z^2 + 2 + z^{-2}$ so if $x = \cos(\frac{2\pi}{5})$, plugging into $(*)$ we get

$4x^2 - 2 + x + 1 = 0$ or $4x^2 + x - 1 = 0$. So x is algebraic and of degree 2 hence constructible.

Use the same trick for $\cos\left(\frac{2\pi}{7}\right)$. If $z = e^{\frac{i2\pi}{7}}$ then

$$u = z + z^{-1} = 2\cos\left(\frac{2\pi}{7}\right)$$

We also have $z^{-3} + z^{-2} + z^{-1} + 1 + z + z^2 + z^3 = 0$

Now $(z + z^{-1})^3 = z^3 + 3z + 3z^{-1} + z^{-3}$ so $u^3 - 3u = z^3 + z^{-3}$

As before, $u^2 - 2 = z^2 + z^{-2}$ so

$$u^3 - 3u + u^2 - 2 + u + 1 = 0 \quad \text{or} \quad u^3 + u^2 - 2u - 1 = 0.$$

This is an irreducible poly so u has degree 3 over $\mathbb{Q}$ and hence so does $\cos\left(\frac{2\pi}{7}\right)$.

In terms of constructibility, since we can construct $\cos\left(\frac{2\pi}{5}\right)$ we can construct a regular pentagon.

Since $\cos\left(\frac{2\pi}{7}\right)$ is not constructible, we cannot construct a regular 7-gon.