Gödel numbering taken up a notch

- Now that we know how to assign Gödel numbers to formulas, we need to understand how we can assign numbers to proofs.
- The basic idea is to apply recursion to the idea of assigning a number to a sequence of symbols; we now need to code sequences of sequences.
- Smith calls this "super Gödel numbering" but I'm going to stick with just coding or Gödel numbering.
- So suppose we have a sequence of sequences of symbols $S_0, S_1, S_2, \ldots, S_n$. We now code this by

$$2^{\ulcorner S_0 \urcorner} 3^{\ulcorner S_1 \urcorner} 5^{\ulcorner S_2 \urcorner} \ldots \pi(i)^{\ulcorner s_i \urcorner} \ldots \pi(n)^{\ulcorner S_n \urcorner}.$$

  Recall that each one of the $\ulcorner S_i \urcorner$'s is a number which is similarly coding the sequence S_i. Again, these numbers are getting absolutely enormous.

Gödel numbers of proofs

- How do we decode a sequence of sequences? That is, given a number *n*, how can we tell if it codes a sequence of sequences?
- As before, we first write out the prime factorization of *n* and read off the exponents of the primes consecutively starting from 2 until the largest prime involved in the factorization. Remember we do this with the exp function and this is all p.r.
- Next we look at each of the exponents and we determine whether this is the Gödel number of a sequence and if so, which one. Again, this entire process is decidable.
- Of course once we have everything decoded as sequences of symbols we revert to what we said in the last instalment of slides: we decidably determine if the sequences are formulas. In this way, we can determine if a given number represents not just a sequence of sequences but is a sequence of formulas.

Gödel numbering of proofs, cont'd

- There is one small problem that we have to address: our proofs aren't sequences of formulas! This is a minor annoyance.
- The issue is that in the proof system we are using, proofs are sequences of sequents. There are other proof systems where proofs are sequences of formulas but what should we do with our system?
- Recall that a sequent looks like $\Sigma \vdash \Theta$ where Σ and Θ are finite sequences of formulas. If we form the conjunction of Σ, i.e. we "and" everything in Σ together, written $\bigwedge \Sigma$ and we form the disjunction of Θ i.e. we "or" everything in Θ together, written $\bigvee \Theta$, then the formula

$$\bigwedge \Sigma \to \bigvee \Theta$$

has the same logical interpretation as the sequent $\Sigma \vdash \Theta$.

Gödel numbering of proofs, cont'd

- So at the expense of turning every sequent into a formula, a very algorithmic procedure, we can think of our proofs as sequences of formulas.
- Now suppose that we have a theory T which is presented to us in a way such that if is possible to decide if a given formula is an axiom of T.
- Then it is entire decidable to determine if a sequence of formulas is a proof starting from T. Now comes the most important bit:
- It is now possible to be given two numbers m and n and to ask the question, which is purely a question of arithmetic, whether the number m codes a proof of the formula n from the theory T. From everything we have said, this relationship is primitive recursive. So there is a Σ_1-formula which we might call $Prov_T(m, n)$ which would mean about the natural numbers, that m codes a proof of the formula n from T.