Effectively enumerate axioms

- We have to go back and fill in a missing step in our proof of the Gödel incompleteness theorem. We assumed that our theory T had an axiomatization which was effectively enumerable and we assumed that this meant we could determine if something was an axiom in a p.r. way i.e. we could find a Σ_1-formula which could express this fact. We didn't quite show this.
- Notice that this is really above and beyond in terms of being careful. There are lots of ways that T could have an effectively enumerable axiomatization that won't require the generality we will now discuss. For instance, Peano arithmetic is technically effectively enumerable but is in fact only finitely many axioms supplemented by a single axiom schema so (assuming you do your homework) we already know how to create a p.r. way of recognizing the axioms of T in cases like this.

Craig's swindle

- Let's imagine that we have a theory T whose axioms can be enumerated in some effective manner. How do we see that Gödel's incompleteness theorem applies in this case?
- Bill Craig made a simple suggestion back in the 1950's to settle this issue. It was called "Craig's swindle" by several logicians at the time.
- We modify our notion of proof just a little. Suppose that P is some procedure for enumerating the axioms of T. Lines in our proofs will now not simply be formulas but they will be pairs (n, φ) where n is some number and φ is a formula. The rules are these: if the pair $(0, \varphi)$ appears in a proof then φ must follow from previous formulas by the application of some rule. If (n, φ) appears and $n > 0$ then φ must be an axiom of T which is enumerated by P in the first n steps of that enumeration.

Craig's swindle cont'd

- Now let's note that if we had a proof in our old system, say $\varphi_0, \ldots, \varphi_n$ is a proof of φ_n from T then it is easy enough to turn it into a proof in the new system. Just put 0 in front of any lines which are derived from earlier lines and if φ_i is an axiom then put the number n beside it representing where it appears in the enumeration determined by P. Of course if we have a new proof then we have an old proof by forgetting about the numbers. So these systems prove the same things.
- To see that the Gödel proof applies now, we need to see that the whole process of Gödel numbering could go through here. I will leave the details to you but just note that the most straightforward way of coding the pair (n, φ) is to code it as

$$2^n 3^{\ulcorner s_0 \urcorner} \ldots \pi(m+1)^{\ulcorner s_m \urcorner}$$

where φ is the sequence $s_0 \ldots s_m$. Essentially we shift the Gödel numbering "over" by 1 and the exponent of 2 codes the number n.

Main Theorem again

- So if we go back and look at the proof of the Gödel theorem, the main sticking point between our assumption of effectively enumerable axiomatizaiton and p.r. axiomatization was the ability to recognize when something was an axiom. Craig allows us a way out of this.
- If we look at the relation $Prov_T(m, n)$ in this new system, it will still hold if m codes a proof of the formula coded by n. This will be p.r. as long as we can recognize when we are dealing with an axiom. So everything comes down to looking at a relation $Ax_T(m)$ which holds if m has the form (n, φ) and φ is an axiom produced by P sometime in the first n steps of this algorithm. But this is a *bounded* process so Ax_T will be Σ_1 and everything works. That is, we have a full proof of

Theorem (Gödel's first incompleteness theorem)

If T is an effectively enumerable theory of arithmetic which proves Q then there is a Π_1-sentence G_T true in N but which is not provable from T.

Wait, what? First? Is there a second? Next time ...