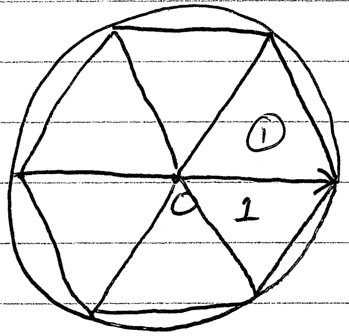
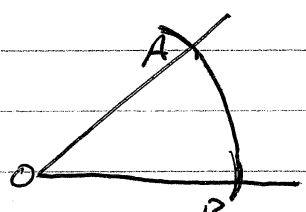
Solutions to Assignment #1

1. In class we constructed an equilateral triangle with a given side. So draw a unit circle and construct an equilateral triangle on a radius to get the triangle ①



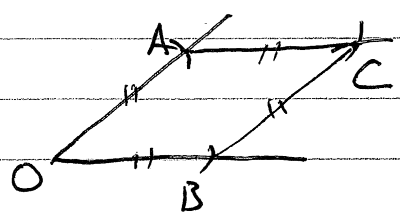
Repeat with each radius constructed to get triangles ② - ⑥ as drawn. The points on the circle form a regular hexagon.

Now one can bisect any angle as follows:

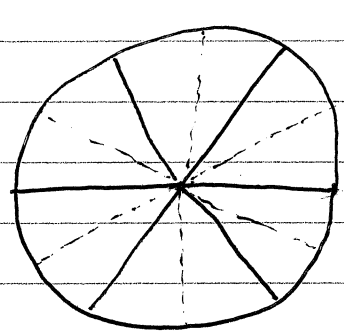


Draw any circle with center O. Label the intersections with the line A and B.

Now with A and B as center, construct the circles with radius $|OA|$ and label the intersection C. The line OC bisects the angle AOB.

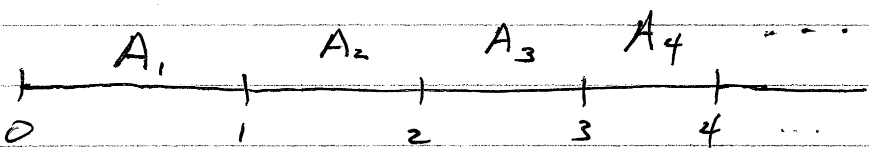


Now to get a regular $2^n \cdot 3$-gon, start with the hexagon constructed above and repeatedly bisect the angles formed at the center.



← 12-gon etc.

2. From class we had constructed sets of reals A_n with the property that $A_n^{(n)}$, the n^{th} derived set, is empty. With these sets we constructed a set A with the property that $A^{(n)} \neq \phi$ for any $n \in \mathbb{N}$ but $\bigcap_{n \in \mathbb{N}} A^{(n)} = \phi$. That set looked like this.



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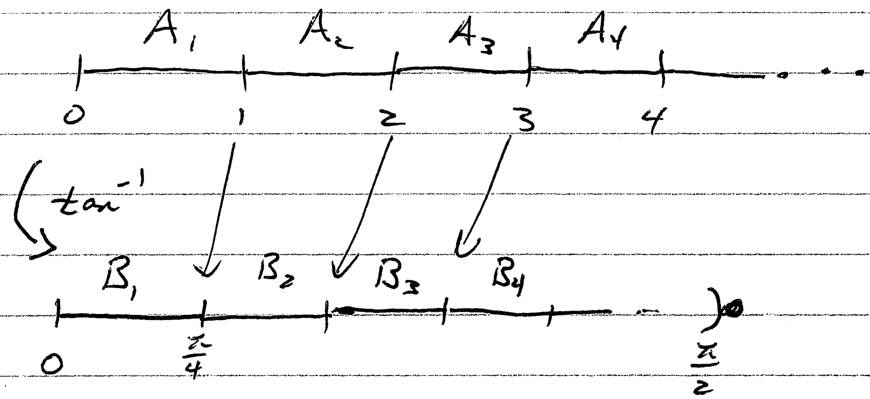
We meant by this that the set A_n may have been shifted and possibly scaled but it still had the property that $A_n^{(n)} = \phi$.

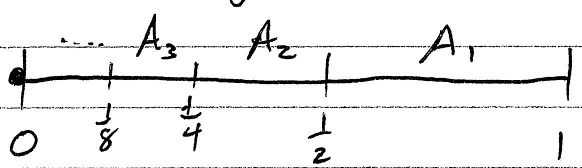
The first proposed solution to this problem was to use $\tan^{-1}$ to map 1 to the interval $[0, \frac{\pi}{2})$.



where B_n is the image of A_n. Since $\tan^{-1}$ is continuous and increasing, $B_n^{(n)} = \phi$ for all $n \in \mathbb{N}$.

Now let $B = B_1 \cup B_2 \cup B_3 \cup \dots \cup \{\frac{\pi}{2}\}$. Since $B_m^{(n)} \neq \phi$ for all $n < m$, $\frac{\pi}{2} \in B^{(n)}$ for all $n \in \mathbb{N}$ so $\bigcap_{n \in \mathbb{N}} B^{(n)} \neq \phi$ and in fact is $\{\frac{\pi}{2}\}$ which will disappear if you derive once more.

Alternatively you could form the set pictured and include 0. The argument is similar.



3. a) $\chi_{\mathbb{N} \setminus A}$ or $\chi_{\bar{A}}$ is $1 - \chi_A$.

b) $\chi_{A \cup B} = \chi_A \cdot \chi_B$

c) $\chi_{A \cap B} = \min \{ 1, \chi_A + \chi_B \}$. We have to say minimum because if $x \notin A$ and $x \notin B$ then $\chi_A(x) + \chi_B(x) = 2$.

d) Let $C = \{ x \in \mathbb{N} : \text{for some } y, (x,y) \in A \}$.

Then $\chi_C(x) = \inf_y \chi_A(x,y)$. Notice that if $(x,y) \in A$ then this inf is 0 otherwise it is 1 which corresponds to the characteristic function of C.

4. Suppose $f : A \to \mathcal{P}(A)$. Then form the set

$B = \{ a \in A : a \notin f(a) \}$. Compare this to what we did for $A = \mathbb{N}$.

Claim: $B \notin \text{rng}(f)$.

If $B \in \text{rng}(f)$ then $B = f(a)$ for some $a \in A$. We ask: Is $a \in B$? If so then $a \in f(a)$ so $a \notin B$ #. If not then $a \notin f(a)$ so $a \in B$. #. This means we contradicted $B \in \text{rng}(f)$ so f is <u>not</u> onto.

Conclusion: There is no onto map from A to $\mathcal{P}(A)$.