Solutions to Assignment 2

1. a) We know that every free abelian group has the form $\coprod_I \mathbb{Z}$. Call this group A. If $a \in A$ and $na = \bar{0}$ then if $a = (a_i : i \in I)$ then $na_i = 0$ for all i. But then $a_i = 0$ for all i and $a = 0$.

Only finitely many a_i's are not zero. If M is the product of all the non-zero a_i's then a is not divisible by any number larger than M. So no element in A is divisible.

b) Suppose we have a map $f : X \setminus Y \to G$ for some abelian group G. Extend f to f' on all of X sending any $y \in Y$ to 0. Since A is free on X, lift f' to a ~~map~~ hom. $F : A \to G$. But $Y \subseteq \ker(F)$ so F defines a map from A/B to G showing A/B is free on $X \setminus Y$.

c) Every element of $\coprod_{\mathbb{N}} \mathbb{Z}$ can be written as a finite linear combination of elements of X. Since $\coprod_{\mathbb{N}} \mathbb{Z}$ is countable, there is then a countable $Y \subseteq X$ so that $\coprod_{\mathbb{N}} \mathbb{Z} \subseteq B$ where B is generated by Y.

d) The map $\prod_{\mathbb{N}} \mathbb{Z} \to S$

$$\langle a_n : n \in \mathbb{N} \rangle \longmapsto \langle n!\, a_n : n \in \mathbb{N} \rangle$$

is a bijection.

e) Choose $a \in S \setminus B$. Since $\coprod_{\mathbb{N}} \mathbb{Z} \subseteq B$

$a \equiv a^k \bmod B$ for some a^k of the form

$(0, \ldots, \underset{\underset{k-1}{\uparrow}}{0}, a^k_k\, a^k_{k+1} \ldots)$ where $l!$ divides a^k_l

for $l \geq k$. So $a + B$ is divisible by k in

$\prod_{\mathbb{N}} \mathbb{Z} / B$ for all k.

f) If $\prod_{\mathbb{N}} \mathbb{Z}$ was free on the generators X, by e) we could choose ctble $Y \subseteq X$ ~~and~~ s.t. $\coprod_{\mathbb{N}} \mathbb{Z} \subseteq B = \langle Y \rangle$. By b), $\prod_{\mathbb{N}} \mathbb{Z}/B$ is free but by e) it contains a divisible element contradicting a).

2. Consider a simple tensor in $R/I \otimes_R R/J$. It is of the form $(r+I)\otimes(s+J) = (1+I)\otimes(rs+J)$ so everything in $R/I \otimes_R R/J$ has the form $(1+I)\otimes(r+J)$ for some $r \in R$.

Now define $f: R/I \times R/J \longrightarrow R/I+J$

$$(r+I,\ s+J) \longmapsto rs + (I+J)$$

To see this is well-defined, suppose $r+I = r'+I$. Then $rs - r's \in I$ so $rs + (I+J) = r's + (I+J)$ for any s. Similarly ~~for~~ when $s+J = s'+J$ then $rs + (I+J) = rs' + (I+J)$.

So we have an R-module hom. $\bar{f}: R/I \otimes_R R/I \twoheadrightarrow R/I+J$. What is it's kernel? $\bar{f}((1+I)\otimes(r+J)) = 0$ implies that $r \in I+J$. But then $r = s+t$ for some $s \in I$, $t \in J$.

$$\begin{aligned}(1+I)\otimes(s+t)+J &= (1+I)\otimes(s+J) + (1+I)\otimes(t+J)\\ &= (s+I)\otimes(1+J) + (1+I)\otimes(t+J)\\ &= 0 + 0 = 0.\end{aligned}$$

So $\bar{f}$ is an isomorphism.

2. a) Since we are given two R-algebras A and B, $A\otimes_R B$ is an R-module which means the additive structure of our proported ring is fine. We need to define mult.

Fix an element of $A \otimes_R B$ say $\sum_{i=1}^{n} a_i \otimes b_i$.

We define a left action of this element on $A \otimes_R B$ by the universal property of tensor products. So define

$$\left(\sum_{i=1}^{n} a_i \otimes b_i\right)(a, b) = \sum_{i=1}^{n} a_i a \otimes b_i b$$ as a function

from $A \times B$ to $A \otimes_R B$. This is easily bilinear and it is R-balanced because R is in the center of both A and B.

So this lifts to an R-module hom that we will also call $\left(\sum_{i=1}^{n} a_i \otimes b_i\right)(\cdot)$ from $A \otimes_R B$ to $A \otimes_R B$.

Notice that left distributivity follows from this being a hom. Right distributivity is almost tautological:

$$\left(\sum_{i=1}^{m}(a_i \otimes b_i) + \sum_{i=m+1}^{n}(a_i \otimes b_i)\right)\left(\sum_{j=1}^{k} c_j \otimes d_j\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{k} a_i c_j \otimes b_i d_j = \sum_{i=1}^{m}\sum_{j=1}^{k} a_i c_j \otimes b_i d_j + \sum_{i=m+1}^{n}\sum_{j=1}^{k} a_i c_j \otimes b_i d_j$$

$$= \left(\sum_{i=1}^{m} a_i \otimes b_i\right)\left(\sum_{j=1}^{k} c_j \otimes d_j\right) + \left(\sum_{i=m+1}^{n} a_i \otimes b_i\right)\left(\sum_{j=1}^{k} c_j \otimes d_j\right).$$

Associativity is a similar annoyance. Notice that

$$\left(\sum_{i=1}^{n} a_i \otimes b_i\right)\left(\left(\sum_{j=1}^{m} c_j \times d_j\right)\left(\sum_{k=1}^{p} e_k \otimes f_k\right)\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} a_i c_j e_k \otimes b_i d_j f_k$$ and the same is true if it is bracketed the other way.

The fact that left multiplication is R-balanced guarantees that R is in the center of $A \otimes_R B$ and $1 \otimes 1$ is clearly the identity.

b) We start by defining an R-module homomorphism from $A \otimes_R B$ to C, using the universal property of $\otimes$:

$$(a, b) \longmapsto f(a)g(b)$$

This is easily bilinear and since R is in the center of C this is R-balanced. ~~We need to show that~~ Let h be the lift of this map to an R-mod. hom from $A \otimes_R B$ to C i.e. $h(a \otimes b) = f(a)g(b)$.

We need to see that h is an R-algebra hom i.e. a ring hom that sends R to R. ($R \hookrightarrow A \otimes_R B$ by $r \mapsto r \otimes 1$). But looking at the generators of $A \otimes_R B$, we have

$$\begin{aligned} h((a \otimes b)(a' \otimes b')) &= h(aa' \otimes bb') = f(aa')g(bb') \\ &= f(a)f(a')g(b)g(b') \\ &= f(a)g(b)f(a')g(b') \\ &= h(a \otimes b)h(a' \otimes b'). \end{aligned}$$

and since the images $f(A)$ and $g(B)$ commute

The general product follows from the linearity of h.

Finally $h(r \otimes 1) = f(r)g(1) = rf(1)g(1) = r$ in C. This h is unique as specified.

4. Define a map from $R[x] \times S$ to $S[x]$ by

$$(p(x), s) \longmapsto sp(x).$$

Bilinearity and R-balanced (because of commutivity) are immediate. So we have an R-module hom.

$$h: R[x] \otimes_R S \to S[x] \quad \text{s.t.} \quad h(p(x) \otimes s) = sp(x).$$

This is clearly onto since $h(x^n \otimes s) = sx^n$ and these terms generated $S[x]$.

Now ~~any~~ if $p(x) = r_0 + r_1 x + \cdots + r_n x^n$ then

$$p(x) \otimes s = 1 \otimes r_0 s + x \otimes r_1 s + \cdots + x^n \otimes r_n s.$$

So all elements of $R[x] \otimes_R S$ are of the form $\sum_{i=0}^{n} x^i \otimes s_i$ for some $s_i \in S$. Now define the map $g: S[x] \to R[x] \otimes_R S$ by $g(s_0 + s_1 x + \cdots + s_n x^n) = \sum_{i=0}^{n} x^i \otimes s_i$. This is easily an R-mod hom and is the inverse of h so h is an isomorphism of R-modules. Use g to see that this is an R-algebra hom:

$$g\left(\left(\sum_{i=0}^{n} s_i x^i\right)\left(\sum_{j=0}^{m} t_j x^j\right)\right) = g\left(\sum_{l=0}^{m+n} \sum_{i+j=l} s_i t_j x^{i+j}\right)$$

$$= \sum_{l=0}^{m+n} \sum_{i+j=l} x^{i+j} \otimes s_i t_j$$

$$= \sum_{\ell=0}^{m+n} \sum_{i+j=\ell} (x^i_\ell \otimes s_i)(x^j \otimes t_j)$$

$$= \sum_{i=0}^{n} (x^i \otimes s_i) \sum_{j=1}^{m} (x^j \otimes t_j)$$

$$= g\left(\sum_{i=0}^{n} s_i x^i\right) g\left(\sum_{j=0}^{m} t_j x^j\right).$$

R is identified with constants in each of $R[x]$ and $S[x]$ and is preserved by h.

5. Since $\mathbb{Q}$ is divisible and all $\mathbb{Z}/p_n\mathbb{Z}$ ~~have~~ are torsion, $\mathbb{Q} \otimes \mathbb{Z}/p_n\mathbb{Z} = 0$ for all n. So we need to see that the RHS is not zero.

It would be great if $\prod_{n \in \mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}$ was divisible but it is not but just barely not i.e. for each m

$\mathbb{Z}/p_n\mathbb{Z}$ is divisible by m except for finitely many p_n (as long as $p_n \nmid m$). So consider

$$A = \prod_{n \in \mathbb{N}} \mathbb{Z}/p_n\mathbb{Z} \Big/ \coprod_{n \in \mathbb{N}} \mathbb{Z}/p_n\mathbb{Z}.$$ and we have a

projection $\pi : \prod_{n \in \mathbb{N}} \mathbb{Z}/p_n\mathbb{Z} \to A$.

Notice that by what was said first, A is divisible. So define

$$\mathbb{Q} \times \prod_{n \in \mathbb{N}} \mathbb{Z}/p_n\mathbb{Z} \to A$$

by $(q, \bar{a}) = q\,\pi(\bar{a})$.

This is bilinear and $\mathbb{Z}$-balanced (you don't have to say this). If one looks at $q = 1$, we see this map is surjective so it lifts to a map

$$h: \mathbb{Q} \otimes \prod_{n \in \mathbb{N}} \mathbb{Z}/p_n\mathbb{Z} \twoheadrightarrow A.$$

Since $A \neq 0$, $\mathbb{Q} \otimes \prod_{n \in \mathbb{N}} \mathbb{Z}/p_n\mathbb{Z} \neq 0$.