1. $Hom_R(M,M)$ is a ring under + and composition (see our first lecture). If M is irreducible then for every $f \in Hom_R(M,M)$, either $Im(f) = 0$ and then $f = 0$ or $Im(f) = M$ and $ker(f) = 0$ so f is invertible so $Hom_R(M,M)$ is a division ring.

(There is no reason to believe composition commutes so it is unlikely to be a field.)

2a) For a field F, any F-vectorspace is both projective and injective.

b) $\mathbb{Z}$ as an abelian group is projective but not injective (it is not divisible).

c) $\mathbb{Q}$ as an abelian group is not projective since free abelian groups contain no divisible elements. $\mathbb{Q}$ is injective since it is divisible.

To see that $\mathbb{Q}$ is flat, suppose $A \xrightarrow{f} B$ is an injection between abelian groups. We described in class $\mathbb{Q} \otimes A$ - all elements of $\mathbb{Q} \otimes A$ can be written as simple tensors $\frac{a}{b} \otimes g$ for $a, b \in \mathbb{Z}$, $b \neq 0$ and $g \in A$. Moreover $\frac{a}{b} \otimes g$ ~~neither~~ $= 0$ iff $a = 0$ or g is torsion.

Now suppose that $\frac{a}{b} \otimes f(g) = 0$ and that $a \neq 0$. Then $f(g)$ is torsion so $n f(g) = 0 = f(ng)$ for some n.

But then $ng = 0$ so $\frac{a}{b} \otimes g = 0$ since g is torsion in A.

d) $\mathbb{Q}/\mathbb{Z}$ is divisible so it is injective as an abelian group but not flat as proved in class.

e) $\mathbb{Q} \oplus \mathbb{Z}$ is not projective (it has divisible elements), it is not injective because it is not divisible and it is flat because $\mathbb{Q}$ and $\mathbb{Z}$ are.

f). $\mathbb{Q}/\mathbb{Z}$ is not flat and $\mathbb{Z}$ is not injective. $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}$ is definitely not injective ($\mathbb{Z}$ is not divisible) and $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}$ is not flat (lift the counter-example for $\mathbb{Q}/\mathbb{Z}$).

3 Since we are looking for a minimal n, we can assume we have only 1 eigenvalue. From our discussion in class, for any matrix A, the degree of the minimal poly. tells us the size of the largest Jordan block when A is in JCF and the number of blocks is the dimension of the eigenspace for our one eigenvalue. So we make the following table:

$n=6$	Size of max block.	# of blocks.	Possibilities. (block sizes)
	6	1	6
	5	2	5+1
	4	2	4+2
	4	3	4+1+1
	3	2	3+3
	3	3	3+2+1
	3	4	3+1+1+1
	2	3	2+2+2
	2	4	2+2+1+1
	2	5	2+1+1+1+1
	1	6	1+⋯+1

Since the max block size and # of blocks uniquely determines the actual block sizes, 6 does not work.

$n=7$ Max block size $=3$ # of blocks $=3$

We could have 3, 3, 1 or 3, 2, 2

So $n=7$ is the minimum. Notice the JCF theorem distinguishes these cases because the # of blocks of size 3 (or 2 or 1) differ in this example.

4 Projective case: Projective modules are submodules of free modules and there is no torsion in a free modules over a p.i.d. A lazy version of the fundamental theorem of f.g. modules over a p.i.d. says that any such module M looks like $\mathrm{Tor}(M) \oplus F$ for a free F, if such an M was projective then it is actually free.

Injective case: Suppose $M \cong R/(a_1) \oplus \cdots \oplus R/(a_k) \oplus R^\ell$
By the Baer criterion, it suffices to see which components are divisible if M is injective.

For a module of the form $R/(a)$ with $a \neq 0$, $1+(a)$ is <u>not</u> divisible by a so this is never injective.

R is injective iff it is a field. So the only f.g. modules over a p.i.d. R which are injective occur when R is a field and then any fin dim v. space is injective.

Flat case: Again, if $M = R/(a_1) \oplus \cdots \oplus R/(a_k) \oplus R^\ell$
it suffices to check component by component.
Free modules are flat, so we need only look at $R/(a)$ for $a \neq 0$.
Consider $\varphi: R \to R$, $\varphi(x) = xa$. Since R is a domain, φ is injective. But $R \otimes R/(a) \xrightarrow{\varphi \otimes 1} R \otimes R/(a)$

and $(\varphi \otimes 1)(r \otimes s + (a)) = (ra \otimes s + (a))$

$$= (r \otimes sa + (a)) = 0.$$

So $\varphi \otimes 1$ is the 0 map which is not injective.

So M is flat iff it is free.

5 We know that A has a Jordan canonical form both over F and over K by assumption. The JCF ~~is still JCF~~ over F is still JCF over K. By the uniqueness of JCF, the two forms must be equivalent.