Assignment 1, Math 4LT3

Due Friday, Jan. 31, by email (please send actual scans or pdfs; no photos)

1. As we discussed in class, the standard manner of capturing graphs in model theory as certain types of binary relations doesn't work if you have multiple edges. I sketched a suggestion of how this could be done. Write this up and include sentences that such a structure must satisfy in order the it captures the class of graphs with multiple edges (and possibly loops).

   **Solution:** For a graph in this generalized sense we could have two disjoint sets $V$ and $E$ which constitute the entire universe of the model. We would also have two relations $R_1$ and $R_2$, both binary, such that if $R_i(x, y)$ holds then $E(x)$ holds as does $V(y)$. Moreover, these relations are functional from the sets $E$ to $V$. They are meant to code the initial and terminal vertices for a given edge.

2. We will show that every field has an algebraically closed extension (which is actually algebraic over the original field).

   (a) Start with any field $F$ and then form the ring $R = F[X_f : f \in Irr(F)]$ where $Irr(F)$ is the set of irreducible polynomials over $F$. Let $I$ be the ideal generated by $\{f(X_f) : f \in Irr(F)\}$. Show that $I$ is a proper ideal of $R$.

   **Solution:** Suppose that $I$ is not a proper ideal. Then we can find $g_1, \ldots, g_n \in R$ and irreducible polynomials $f_1, \ldots, f_n$ such that

   $$g_1 f_1(X_{f_1}) + \ldots + g_n f_n(X_{f_n}) = 1.$$

   Now choose some field $K$ extending $F$ in which $f_1, \ldots, f_n$ have solutions - this is a finite construction adjoining a root of each polynomial one by one. Now evaluate the displayed expression at the roots in $K$ to get the absurdity $0 = 1$. So $I$ is a proper ideal.

   (b) Choose $M \supset I$ a maximal ideal in $R$ and show that $R/M$ can be thought of as an algebraic field extension of $F$ in which every irreducible polynomial over $F$ has a solution.

   **Solution:** As $M$ is a maximal ideal and $R$ is a commutative ring, $R/M$ is a field and moreover, $F$ naturally embeds into $R/M$ so we have a field extension of $F$. Every irreducible polynomial $f$ over $F$ is satisfied by the image of $X_f$ in $R/M$ since $f(X_f) \in I$ and this means that $R/M$ is generated by algebraic elements over $F$ and so $R/M$ is an algebraic field extension of $F$.

(c) Now form a chain $F = F_0 \subset F_1 \subset F_2 \ldots F_n \subset \ldots$ such that $F_{n+1}$ is an algebraic extension of $F_n$ which contains a solution for every irreducible polynomial over $F_n$. Conclude that $\bigcup F_n$ is an algebraically closed field which is algebraic over $F$.

**Solution:** Since each $F_{n+1}$ is algebraic over $F_n$, we conclude that $K = \bigcup F_n$ is algebraic over $F$. Moreover, since any polynomial over $K$ contains only finitely many coefficients, such a polynomial is also a polynomial over $F_n$ for some $n$ and has a solution in $F_{n+1}$ so $K$ is an algebraically closed field.

(d) It turns out that if $F \subset K$ are fields such that $K$ is algebraically closed and algebraic over $K$ then $K$ is unique up to isomorphism over $F$ and is called the algebraic closure of $F$.

3. We wrote out sentences in the language of fields which were satisfied by an algebraically closed field of characteristic $p$ where $p$ is some prime or 0. We want to show that this set of sentences is complete.

(a) We need to define what is called a transcendental set of elements inside a field: $X \subset F$ is called transcendental if for all $x \in X$, $x$ is not algebraic over $X \setminus \{x\}$. Show that if $X \subset F$ is a maximal transcendental set then $F$ is algebraic over $X$.

**Solution:** If $F$ is not algebraic over $X$ then choose some $x \in F$ which is not algebraic over $X$. But then $X \cup \{x\}$ is transcendental and contradicts the maximality of $X$.

(b) Suppose that $X \subset F$ and $Y \subset G$ are maximal transcendental sets and $F$ and $G$ are algebraically closed of characteristic $p$. Show that if $f : X \to Y$ is a bijection then $f$ can be extended to an isomorphism from $F$ to $G$.

**Solution:** As I said in class, we use heavily the fact that algebraic closures are unique up to isomorphism. Since $X$ and $Y$ are bijective and transcendental, it is easy to show that the bijection $f$ lifts to a bijective homomorphism $\bar{f}$ between the two subfields generated by $X$ and $Y$. But as $F$ is the algebraic closure of $\langle X\rangle$ and $G$ is the algebraic closure of $\langle Y\rangle$, the uniqueness of the algebraic closure guarantees that $\bar{f}$ lifts to an isomorphism from $F$ to $G$.

(c) Now show that if $F$ and $G$ are algebraically closed fields of characteristic $p$ then there are $F'$ and $G'$ of the same uncountable cardinality such that $F \prec F'$ and $G \prec G'$. Conclude that $F' \cong G'$ and so $F$ and $G$ have the same theory.

**Solution:** Choose some cardinal $\lambda > |F|, |G|$. By the upward Lowenheim-Skolem theorem, there are $F'$ and $G'$ as described both of size $\lambda$. Now if $X$ is a transcendence basis for $F'$ and $Y$ is a transcendence basis for $G'$ then $|X| = |F'| = |G'| = |Y|$ and so by what we just said in the previous part, $F' \cong G'$. This guarantees that $F \equiv G$.

4. Prove the Łoś theorem.

**Solution:** Suppose that $M_i$ are $L$-structures for each $i \in I$, $U$ is an ultrafilter on $I$ and $M = \prod_U M_i$. We want to show that if $\varphi(x_1, \ldots, x_n)$ is an $L$-formula and $m^1, \ldots, m^n \in M$ with $m^j = (m_i^j : i \in I)_U$ for $j = 1, \ldots, n$ then

$$M \models \varphi(m^1, \ldots, m^n) \text{ iff } \{i \in I : M_i \models \varphi(m_i^1, \ldots, m_i^n) \in U.$$

We do this by induction on the formation of the formula $\varphi$. Atomic formulas are immediate by the definition of the ultraproduct and the connectives are straightforward using the properties of ultrafilters (note that we need the ultrafilter property for negation). This leaves us with checking a formula of the form $\exists y \varphi(y, x_1, \ldots, x_n)$.

So assume that $M \models \exists y \varphi(y, m^1, \ldots, m^n)$ and choose $m \in M$ such that $M \models \varphi(m, m^1, \ldots, m^n)$. By induction we have

$$\{i \in I : M_i \models \varphi(m_i, m_i^1, \ldots, m_i^n)\} \in U.$$

But then $\{i \in I : M_i \models \exists y \varphi(y, m_i^1, \ldots, m_i^n)\} \in U$. which is what we want.

In the other direction, if $\{i \in I : M_i \models \exists y \varphi(y, m_i^1, \ldots, m_i^n)\} \in U$ then for each $i \in I$, choose $m_i$ such that $M_i \models \varphi(m_i, m_i^1, \ldots, m_i^n)$ if possible and let it be anything you want if this is not possible. By induction $M \models \varphi(m, m^1, \ldots, m^n)$ where $m = (m_i : i \in I)$ and so $M \models \exists y \varphi(y, m^1, \ldots, m^n)$.

5. Ultraproducts are used in other areas of mathematics; here is a fairly typical example.

   (a) Suppose that $r_i \in R$ for all $i \in I$ is a bounded family of real numbers. Fix an ultrafilter $U$ on $I$. Prove that there is a unique number $r$ such that for every $\epsilon > 0$, $\{i \in I : |r - r_i| < \epsilon\} \in U$. We call $r$ the ultralimit of the sequence $r_i$ along the ultrafilter $U$ and write $r = \lim_{i \to U} r_i$.

   **Solution:** We first handle the uniqueness. Suppose that $r$ and $s$ both satisfy the condition of being the ultralimit of the $r_i$'s. Let $\epsilon = \frac{|r-s|}{2}$. We would have $X = \{i \in I : |r - r_i| < \epsilon\} \in U$ and $Y = \{i \in I : |s - r_i| < \epsilon\} \in U$. But if $i \in X \cap Y$ then

   $$|r - s| = 2\epsilon > |r - r_i| + |r_i - s| \geq |r - s|$$

   which is a contradiction.

   By rescaling the sequence, we can assume that all the $r_i$'s are in the interval $[0, 1]$. We now define a decresing sequence of intervals $I_n$ for $n \in N$ with $I_0 = [0, 1]$. The inductive condition is that $\{i \in I : r_i \in I_n\} \in U$. Suppose we have determined $I_n = [a, b]$ and consider the two intervals $[a, c]$ and $[c, b]$ where $c = \frac{a+b}{2}$. If $\{i \in I : r_i \in [a, c]\} \in U$ then let $I_{n+1} = [a, c]$. Otherwise let $I_{n+1} = [c, b]$. It is easy to show that if $X \cup Y \in U$ then at least one of $X$ or $Y$ is in $U$. It follows that if $I_{n+1}$ is not $[a, c]$ then $\{i \in I : r_i \in [b, c]\} \in U$. It is also straightforward to show that the length of $I_n$ is $\frac{1}{2^n}$. So the intersection of the intervals $I_n$ is $r$ for some $r \in [0, 1]$. We now show that $r$ is the ultralimit of the $r_i$'s. Choose $\epsilon > 0$ and fix $n$ such that $\frac{1}{2^n} < \epsilon$. Then since $r \in I_n$, if $r_i \in I_n$ then $|r - r_i| \leq \frac{1}{2^n} < \epsilon$. So $\{i \in I : |r - r_i| < \epsilon\} \in U$ which demonstrates that $r$ is the ultralimit.

   (b) Suppose that $(X_i, d_i)$ is a metric space for each $i \in I$ and fix a point $a_i \in X_i$. Consider the set $X$ be the set of all $I$-indexed sequences $\{\langle x_i : i \in I\rangle$ such that $x_i \in X_i$ for all $i \in I$ and there is some $M$ such that $d_i(x_i, a_i) \leq M$ for all $i \in I$. Fix an ultrafilter $U$ on $I$ and define $d$ on $X$ by

$$d(\bar{x}, \bar{y}) = \lim_{i \to U} d_i(x_i, y_i)$$

Show that $d$ is a pseudo-metric on $X$; the quotient of $X$ by this pseudo-metric is the metric ultraproduct of the $X_i$'s.

**Solution** Possibly the only thing you really need to check here is that the ultralimit is well-defined. But for each $\bar{x} \in X$, there is some $M$ such that $d_i(a_i, x_i) \leq M$ for all $i$. So if $\bar{x}, \bar{y} \in X$ then there are $M$ and $N$ such that $d(a_i, x_i) \leq M$ and $d(a_i, y_i) \leq N$. So by the triangle inequality $d_i(x_i, y_i) \leq M + N$ which means that the sequence $d_i(x_i, y_i)$ is bounded and the ultralimit $d(\bar{x}, \bar{y})$ is well-defined. Now to see that $d$ is a pseudo-metric, it is clear that $d(\bar{x}, \bar{x}) = 0$ for all $\bar{x} \in X$. Symmetry is also similarly easy. If $\bar{x}, \bar{y}$ and $\bar{z}$ are all in $X$ then we know that for each $i$,

$$d_i(x_i, y_i) + d_i(y_I, z_i) \geq d_i(x_i, z_i).$$

It follows easily that taking ultralimits of both sides gives
$d(\bar{x}, \bar{y}) + d(\bar{y}, \bar{z}) \geq d(\bar{x}, \bar{z})$.