Assignment 1, Math 711
Due Feb. 3, in class

1. Suppose that K is an algebraically closed field and $X \subseteq K$. We say that X is algebraically independent if for all $x \in X$, x is not algebraic over $X \setminus \{x\}$. A maximal algebraically independent subset of K is called a transcendence basis for K.

   (a) Prove that every algebraically closed field contains a transcendence basis.

   (b) Prove that any algebraically closed field is the algebraic closure of any of its transcendence bases.

   (c) Show that if K is algebraically closed and X is a transcendence basis then $|K| = |X| + \aleph_0$.

   (d) Show that if K and L are algebraically closed fields of the same uncountable cardinality and characteristic then $K \cong L$.

   (e) Conclude that the theories ACF_p for primes p and ACF_0 are complete theories.

2. Prove Łoś' Theorem.

3. Prove the following enhanced version of the elementarily of the diagonal embedding into an ultrapower: Suppose we have two sets I and J, a function $f : J \to I$ and an ultrafilter U on J.

   (a) Prove that $f[U] = \{Y \subseteq I : \text{ for some } X \in U, f(X) \subseteq Y\}$ is an ultrafilter on I.

   (b) Now suppose that M_i is an L-structure for all $i \in I$. Define a map from $\prod_{i \in I} M_i / f[U]$ to $\prod_{j \in J} M_{f(j)} / U$ by

   $$\langle m_i : i \in I \rangle / f[U] \mapsto \langle m_{f(j)} : j \in J \rangle / U$$

   Prove that this is an elementary map.

   (c) Consider the case where I is a singleton and J is any set. Let f be the only function from J to I and deduce that this construction is the diagonal embedding into the ultrapower for any ultrafilter on J.

4. Ultraproducts are used in other areas of mathematics; here is a fairly typical example.

   (a) Suppose that $r_i \in R$ for all $i \in I$ is a bounded family of real numbers. Fix an ultrafilter U on I. Prove that there is a unique number r such that for every $\epsilon > 0$, $\{i \in I : |r - r_i| < \epsilon\} \in U$. We call r the ultralimit of the sequence r_i along the ultrafilter U and write $r = \lim_{i \to U} r_i$.

   (b) Suppose that (X_i, d_i) is a metric space for each $i \in I$ and fix a point $a_i \in X_i$. Consider the set X be the set of all I-indexed sequences $\{\langle x_i : i \in I\rangle$ such that $x_i \in X_i$ for all $i \in I$ and there is some M such that $d_i(x_i, a_i) \leq M$ for all $i \in I$. Fix an ultrafilter U on I and define d on X by

$$d(\bar{x}, \bar{y}) = \lim_{i \to U} d_i(x_i, y_i)$$

   Show that d is a pseudo-metric on X; the quotient of X by this pseudo-metric is the metric ultraproduct of the X_i's.