Assignment 2, Math 711
Due March 3

1. We call a class of L-structures elementary if it is the class of models of some set of L-sentences. Prove that a class is elementary iff it is closed under isomorphisms, elementary submodels and ultraproducts.

2. Prove that an elementary class is closed under unions of chains iff it has $\forall\exists$-axioms i.e. sentences of which begin with a block of universal quantifiers followed by a block of existential quantifiers and then a quantifier free formula. Here is an extended hint: suppose that C is the class of models of the theory T and C is closed under unions of chains. Let T_0 be all those $\forall\exists$-sentences φ such that $T \models \varphi$. We will construct a sequence of models

$$M_0 \subseteq N_0 \subseteq M_1 \subseteq N_1 \subseteq ...$$

such that $M_i \models T_0$, $N_i \models T$ and $M_i \prec M_{i+1}$ for all i.

Claim: if we can do this we are done - why?

The basis step in this construction is then to take $M_0 \models T_0$ and find N_0 and M_1 as above and then repeat as needed. Define

$$Diag_\forall(M_0) = \{\forall\bar{y}\varphi(\bar{m},\bar{y}) : \varphi \text{ is qff }, \bar{m} \in M \text{ and } M \models \forall\bar{y}\varphi(\bar{m},\bar{y})\}$$

Show that $T \cup Diag_\forall(M_0)$ is satisfiable in the language L_{M_0}. Choose N_0, a model of this theory. Now show that $Elem(M_0) \cup Diag(N_0)$ is satisfiable to get M_1.

3. Describe the type space in one variable for the theory of $(Q, <)$ with a constant r for every $r \in Q$ interpreted as itself. Do the same for the theory of an algebraically closed field K with a constant for every $a \in K$ again interpreted as itself.

4. Use EF-games to determine all the complete theories of a single equivalence relation.