Assignment 3, Math 711
Due March 24, in class

1. Prove that if M is a countable model in a countable language then M can be elementarily extended to a countable homogeneous model.

2. Show that if M_n are countable models for $n \in N$ in a countable language and U is a non-principal ultrafilter on N then the ultraproduct $\prod_{n \in N} M_n/U$ is $\aleph_1$-saturated.

3. Suppose we have a language with a single binary relation symbol $<$ and constants c_n for all $n \in N$. Consider the theory T in this language which says that $<$ is a dense linear order without endpoints and that $c_i < c_j$ whenever $i < j$. This theory has three countable models up to isomorphism. Describe them.

4. Let $\mathcal{C}$ be the class of all finite graphs i.e. finite sets together with an irreflexive, symmetric, binary relation called the edge relation. Show that $\mathcal{C}$ is closed under submodels and amalgamation i.e. if $A, B, C \in \mathcal{C}$ and $f : A \to B$ and $g : A \to C$ are embeddings then there is $D \in \mathcal{C}$ together with embeddings $i : B \to D$ and $j : C \to D$ such that $jg = if$. Using these properties and the fact that there are only finitely many graphs up to isomorphism of size n to construct a countable graph G with the property that all graphs in $\mathcal{C}$ embed into G and if A is a finite subgraph of G and $A \subseteq B \in \mathcal{C}$ then there is an embedding of B into G which is the identity on A. Prove that any graph with these properties is unique up to isomorphism and determine axioms for the theory of this graph.