Solutions to Assignment 1

1 a) Zorn's lemma - in more detail, subsets of K which are algebraically independent are closed under unions of chains and so Zorn's lemma can be applied to generate a maximal alg. indep. set.

b) If $X \subseteq K$ is a trans. basis then the algebraic closure of X or more formally, the alg. closure of the field generated by X must be all of K. If not, there is $x \in K$ which is not alg over X then $X \cup \{x\}$ is alg. indep.

c) Every element of K is algebraic over $K_0(X)$ where K_0 is the prime field. There are at most $|X| + \aleph_0$ many poly. over $K_0(X)$ and each has only finitely many roots so $|K| \leq |X| + \aleph_0$. K cannot be finite

$\left(\prod_{a \in K} (x-a) + 1 \text{ would have no root} \right)$ so $|K| = |X| + \aleph_0$

d) If K and L are of unctble cardinality then from c) $|K| = |X|$ and $|L| = |Y|$ where X is a transcendence basis for K and Y is one for L. If K_0, L_0 are the resp. prime fields then $K_0 \cong L_0$ since K and L have the same characteristic. Pick any bijection $f: X \to Y$.

Claim: $\varphi: K_0(X) \to L_0(Y)$ determined by $\varphi(x) = f(x)$ and ~~K_0~~ $\varphi: K_0 \xrightarrow{\sim} L_0$ is a well-defined isomorphism.

Pf/ It suffices to see that if $\varphi(p(\bar{x})) = 0$ then $p(\bar{x}) \equiv 0$. But for $\bar{x} \in X$, $\bar{x}$ is alg. indep so there is no non-trivial polynomial p s.t. $p(\bar{x}) = 0$. So $p \equiv 0$.

Now φ lifts to a map from $\overline{K_0(X)} = K$ to $\overline{L_0(Y)} = L$.

e) If $M, N \models ACF_p$ then there is $\lambda > |M|, |N|$ and fields M', N', $|M'| = |N'| = \lambda$ with $M \prec M'$ and $N \prec N'$ (by upward LS). But then $M' \cong N'$ by d) so

$M \equiv M' \cong N' \equiv N$ which implies that ACF_p is complete.

2. Suppose M_i for $i \in I$ are $\mathcal{L}$-structures, $\mathcal{U}$ is an ultrafilter on I and $M = \prod_{i \in I} M_i / \mathcal{U}$.

We wish to prove that

$$M \models \varphi(\bar{a}/\mathcal{U}) \quad \text{iff} \quad \{ i \in I : M_i \models \varphi(\bar{a}_i) \} \in \mathcal{U}$$

by induction on formulas.

Atomic formulas follow by the definition of the ultraproduct. Let's consider the basic connectives $\wedge$ and $\neg$: $\wedge$ follows since filters are closed under intersection. For $\neg$

$$\begin{aligned} M \models \neg\varphi(\bar{a}/\mathcal{U}) \quad &\text{iff} \quad M \not\models \varphi(\bar{a}/\mathcal{U}) \\ &\text{iff} \quad \{ i \in I : M_i \models \varphi(\bar{a}_i) \} \notin \mathcal{U} \text{ by ind.} \\ &\text{iff} \quad \{ i \in I : M_i \not\models \varphi(\bar{a}_i) \} \in \mathcal{U} \text{ since } \mathcal{U} \text{ is an ultrafilter.} \end{aligned}$$

Now the existential quantifier:

$M \models \exists x\, \varphi(x, \bar{a}/\mathcal{U})$ iff for some $b/\mathcal{U}$
$M \models \varphi(b/\mathcal{U}, \bar{a}/\mathcal{U})$
iff $\{i \in I : M \models \varphi(b_i, \bar{a}_i)\} \in \mathcal{U}$
by ind.

Now $\{i \in I : M \models \varphi(b_i, \bar{a}_i)\} \subseteq \{i \in I : M \models \exists x\, \varphi(x, \bar{a}_i)\}$ so if the first is in $\mathcal{U}$ then so is the second.

What if $\{i \in I : M \models \exists x\, \varphi(x, \bar{a}_i)\} \in \mathcal{U}$? Define a sequence b by $b_i \in M_i$ with $M_i \models \varphi(b_i, \bar{a}_i)$ if

$M_i \models \exists x\, \varphi(x, \bar{a}_i)$ and anything otherwise.

Now $M \models \varphi(b/\mathcal{U}, \bar{a}/\mathcal{U})$ since $\{i \in I : M_i \models \varphi(b_i, \bar{a}_i)\} \in \mathcal{U}$ so $M \models \exists x\, \varphi(x, \bar{a}/\mathcal{U})$.

3 a) $f[\mathcal{U}]$ is clearly closed upwards. If $Y, Z \in f[\mathcal{U}]$ then there are $V, X \in \mathcal{U}$ s.t. $f(V) \subseteq Y$, $f(X) \subseteq Z$ so

$f(V \cap X) \subseteq f(V) \cap f(X) \subseteq Y \cap Z$ so $Y \cap Z \in f[\mathcal{U}]$

Now for any $Y \subseteq I$, consider $X = \{x \in J : f(x) \in Y\}$. If $X \in \mathcal{U}$ then $f(X) \subseteq Y$ and so $Y \in f[\mathcal{U}]$. If $X \notin \mathcal{U}$ then $X^c \in \mathcal{U}$ and $f(X^c) \subseteq Y^c$ so $Y^c \in f[\mathcal{U}]$.

b) Probably the hardest thing here is to see that this map is well-defined.

Suppose that $\langle m_i : i \in I \rangle$ and $\langle m_i' : i \in I \rangle$ are $f[\mathcal{U}]$-equivalent. We want to show $\{ j \in J : m_{f(j)} = m'_{f(j)} \}$ is in $\mathcal{U}$. By assumption $Y = \{ i \in I : m_i = m_i' \} \in f[\mathcal{U}]$

So there is $X \in \mathcal{U}$ s.t. $f(X) \subseteq Y$. But if $j \in X$ $f(j) \in Y$ and so $m_{f(j)} = m'_{f(j)}$. This means

$$X \subseteq \{ j \in J : m_{f(j)} = m'_{f(j)} \} \in \mathcal{U}.$$

Now to see elementarity: Define the map

$$\langle m_i : i \in I \rangle / f[\mathcal{U}] \longmapsto \langle m_{f(j)} : j \in J \rangle / \mathcal{U}$$

to be g. Fix $\bar{m} = \langle \bar{m}_i : i \in I \rangle / f[\mathcal{U}] \in \prod_{i \in I} M_i / f[\mathcal{U}]$

and we wish to see that

$$\prod_{i \in I} M_i / f[\mathcal{U}] \models \varphi(\bar{m}) \quad \text{iff} \quad \prod_{j \in J} M_{f(j)} / \mathcal{U} \models \varphi(g(\bar{m}))$$

So $\prod_{i \in I} M_i / f[\mathcal{U}] \models \varphi(\bar{m})$ iff $\{ i \in I : M_i \models \varphi(\bar{m}_i) \} \in f[\mathcal{U}]$

iff there is $X \in \mathcal{U}$ s.t.

$$f(X) \subseteq \{ i \in I : M_i \models \varphi(\bar{m}_i) \}.$$

If the latter holds then $X \subseteq \{ j \in J : M_{f(j)} \models \varphi(\bar{m}_{f(j)}) \} \in \mathcal{U}$

and so $\prod_{j \in J} M_{f(j)} / \mathcal{U} \models \varphi(g(\bar{m}))$.

In the other direction, if $\prod_{j\in J} M_{f(j)}/\mathcal{U} \models \varphi(g(\bar{m}))$

then $X = \{ j \in J : M_{f(j)} \models \varphi(\bar{m}_{f(j)}) \} \in \mathcal{U}$.

So $f(X) \subseteq \{ i \in I : M_i \models \varphi(\bar{m}_i) \}$ which puts it in $f[\mathcal{U}]$.

c) If $I = \{*\}$, J is any set and $\mathcal{U}$ any ultrafilter on J then the construction from b) is

$$M \hookrightarrow \prod_{j\in J} M_j/\mathcal{U} \quad \text{given by} \quad m \mapsto \langle m : j \in J \rangle$$

the diagonal embedding.

4 a) Suppose that all r_i for $i \in I$ are in the interval $[a,b]$. Toward a contradiction, suppose that for every $r \in [a,b]$ there is an ε_r s.t.

$$X_r = \{ i \in I : |r_i - r| < \varepsilon_r \} \notin \mathcal{U}.$$

The sets $(r-\varepsilon_r, r+\varepsilon_r)$ form an open cover of $[a,b]$ and so there are finitely many r's ; $r^1 \ldots r^n$ s.t.

$$(r^i - \varepsilon_{r^i}, r^i + \varepsilon_{r^i}) \quad \text{cover } [a,b].$$

But $\{ i \in I : \text{for some } j = 1, \ldots, n \ \ |r^j - r_i| < \varepsilon_{r^j} \} = I$ and $\mathcal{U}$ is an ultraproduct so one of the sets

$X_{r^i} \in \mathcal{U}$. The r is unique by considering two such r's, r^1 and r^2 and looking at $\varepsilon = \dfrac{|r^1 - r^2|}{2}$

b) $\{ d_i(x_i, y_i) : i \in I \}$ is bounded since there is some M, N with $d_i(x_i, a_i) \leq M$ and

$d(y_i, a_i) \leq N$ for all $i \in I$. So $d(x_i, y_i) \leq M + N$.

So $d(\bar{x}, \bar{y})$ is well-defined from a).

$d(\bar{x}, \bar{x}) = 0$ and $d(\bar{x}, \bar{y}) = d(\bar{y}, \bar{x})$ clearly.

For the triangle inequality, for each i

$$d(x_i, z_i) \leq d_i(x_i, y_i) + d_i(y_i, z_i)$$

and taking ultralimits on both sides we get

$$d(\bar{x}, \bar{z}) \leq d(\bar{x}, \bar{y}) + d(\bar{y}, \bar{z}).$$