Solutions to Assignment #2

1. Suppose that $\mathcal{C}$ is the class of models of a first order theory T. If $M \in \mathcal{C}$ and $N \cong M$ then $N \models T$. If $N \prec M$ then also $N \models T$.
If $M_i \in \mathcal{C}$ for $i \in I$ and $\mathcal{U}$ is an ultrafilter on I then since each $M_i \models T$, by Łoś, $\prod M_i/\mathcal{U} \models T$ and so $\mathcal{C}$ is closed under ultraproducts.

Now suppose $\mathcal{C}$ is a class of $\mathcal{L}$-structures closed under isomorphisms, elementary submodels and ultraproducts. Let $T = \{ \varphi : \varphi$ is an $\mathcal{L}$-sentence and $M \models \varphi$ for all $M \in \mathcal{C} \}$.

We wish to show that $\mathcal{C}$ is the class of models of T. To do this, we choose $M \models T$ and show that ~~$M \xrightarrow{elem}$~~ M embeds elementarily into an ultraproduct

of structures in $\mathcal{C}$. To this end, consider

$T \cup \mathrm{Elem}(M)$. ~~This is satisfiable. We will show it is finitely satisfiable. Elem(M) is closed conjunctions so A $T \cup \mathrm{Elem}(M)$ is not via~~

For every finite $\Sigma \subseteq \mathrm{Elem}(M)$, I claim there is $M_\Sigma \in \mathcal{C}$ in which we can interpret the constants in Σ s.t. $M_\Sigma \models \Sigma$. If not, then $T \models \neg \exists \bar{x} \bigwedge_{\varphi \in \Sigma} \varphi$

but then $M \models \neg \exists \bar{x} \bigwedge_{\varphi \in \Sigma} \varphi$ which is a contradiction.

So pick a non-principal ultrafilter $\mathcal{U}$ and

$I = \{ \Sigma \subseteq_{\cancel{fin}} Elem(M) : \Sigma \text{ is finite} \}$ and look at

$\prod_{\Sigma \in I} M_\Sigma / \mathcal{U}$. $M \hookrightarrow \prod_{\Sigma \in I} M_\Sigma / \mathcal{U}$ via the map

which sends m to its interpretation in $\cancel{M_\Sigma}$ $\prod_{\Sigma \in I} M_\Sigma / \mathcal{U}$

which is defined almost everywhere and we get $\prod_{\Sigma \in I} M_\Sigma / \mathcal{U} \models Elem(M)$

so this embedding is elementary. So $M \in \mathcal{E}$.

2. Following the hint: Suppose that $\mathcal{C} = Mod(T)$, the class of models of T and $T_0 = \{ \varphi : \varphi \text{ is } \forall\exists \text{ and } M \models \varphi \text{ for all } M \in \mathcal{C} \}$.

Let $M_0 \models T_0$ and define $At_\forall(M_0) = \{ \varphi(\bar{m}) : \varphi \text{ is universal and } M_0 \models \varphi(\bar{m}) \}$

Claim: $T \cup At_\forall(M_0)$ is satisfiable.

Again, since $At_\forall(M_0)$ is, up to equivalence, closed under conjunction, if $T \cup At_\forall(M_0)$ is not satisfiable then there is $\varphi(\bar{m}) \in At_\forall(M_0)$ s.t.

$$T \models \neg\varphi(\bar{m}) \quad \text{and so} \quad T \models \forall\bar{x}\, \neg\varphi(\bar{x})$$

However, φ is universal, say $\varphi = \forall\bar{y}\, \psi(\bar{x}, \bar{y})$ so $T \models \forall\bar{x}\, \exists\bar{y}\, \neg\psi(\bar{x}, \bar{y})$ and so $\forall\bar{x}\, \exists\bar{y}\, \neg\psi(\bar{x}, \bar{y}) \in T_0$.

But this contradicts the fact that $M_0 \models \varphi(\bar{m})$.

So choose $N_0 \models T \cup At_\forall(M_0)$. WMA $M_0 \subseteq N_0$

Claim: $Elem(M_0) \cup At(N_0)$ is satisfiable.

If not, then there is $\varphi(\bar{n}, \bar{m}) \in At(N_0)$ s.t.

$Elem(M_0) \models \neg\varphi(\bar{n}, \bar{m})$ and so $Elem(M_0) \models \forall\bar{y}\, \neg\varphi(\bar{y}, \bar{m})$.
So $\forall\bar{y}\, \neg\varphi(\bar{y}, \bar{m}) \in At_\forall(M_0)$ which contradicts $N_0 \models At_\forall(M_0)$.

So choose $M_1 \supseteq N_0$ and such that $M_0 \prec M_1$.

Now repeat this process with M_1 in place of M_0 to get a chain:

$$M_0 \subseteq N_0 \subseteq M_1 \subseteq N_1 \subseteq M_2 \subseteq \cdots \quad \text{s.t.}$$

① $M_i \prec M_{i+1}$ for all i and

② $N_i \in \mathcal{C}$

By assumption, $\bigcup N_i \in \mathcal{C}$ and $\bigcup N_i = \bigcup M_i$ with $M_0 \prec \bigcup M_i$ so $M_0 \in \mathcal{C}$.

In the other direction, if T is $\forall\exists$-axiomatizable and M_i for $i \in I$ is a chain of models of T then for any $\forall\exists$-axiom, say $\forall \bar{x} \exists \bar{y}\, \varphi(\bar{x}, \bar{y})$, to check this in $\bigcup M_i$, pick $\bar{m} \in \bigcup M_i$ so $\bar{m} \in M_i$ for some i. Since $M_i \models T$, there is $\bar{n} \in M_i$ s.t. $M_i \models \varphi(\bar{m}, \bar{n})$. Since φ is q.f.f., $\bigcup M_i \models \varphi(\bar{m}, \bar{n})$. So $\bigcup M_i \models \forall x \exists y\, \varphi$ and so $\bigcup M_i \models T$.

3 a) By quantifier elimination in the theory of dense linear orders, all types over $(\mathbb{Q}, <)$ are determined by the formulas $x<r$, $x=r$ and $x>r$ for $r \in \mathbb{Q}$ that they contain. Since $<$ is an order on $\mathbb{Q}$, for any 1-type p, $\{r \in \mathbb{Q} : x<r \in p\}$ and $\{r \in \mathbb{Q} : x>r \in p\}$ forms a cut in $\mathbb{Q}$.

If either of these two sets is empty then we have respectively the types of an infinite positive and infinite negative element.

If both sets are not empty, the cut could be irrational and there is one such type for every $r \in \mathbb{R} \setminus \mathbb{Q}$.

If the cut is rational, there are three possibilities:

1. the isolated possibility that $x = r$ for some $r \in \mathbb{Q}$.

2. infinitesimally close to $r \in \mathbb{Q}$ on the left i.e. $x < r$ but $x > s$ for all $s < r$, $s \in \mathbb{Q}$.

or 3. inf. close to $r \in \mathbb{Q}$ on the right i.e. $x > r$ but $x < s$ for all $s > r$, $s \in \mathbb{Q}$.

b) If p is a 1-type over an alg. closed field K then one possibility is that $x = a \in p$ for some $a \in K$. If this is not true then since we have quant elim. for $Th(K)$, p is determined by the atomic formulas it satisfies. In this case, these are polynomial equations $p(x) = 0$ for p over K. But K is alg. closed so such a p factors completely over K. So if $x \neq a$ for all $a \in K$ then $p(x) \neq 0$ for all poly. p over K. So there is one non-isolated, transcendental type.

4. We are considering all completions of the theory of a single equivalence relation E. Define two functions determined by a model $\mathcal{M} = (M, E)$:

$$f_{\mathcal{M}} : \mathbb{N}^{\geq 1} \to \mathbb{N} \cup \{\infty\} \quad \text{and} \quad g_{\mathcal{M}} : \mathbb{N}^{\geq 1} \to \mathbb{N} \cup \{\infty\}$$

where $f_{\mathcal{M}}(n)$ = the number of equivalence classes in $\mathcal{M}$ with exactly n elements.

and $g_{\mathcal{M}}(n)$ = the number of equiv. classes in $\mathcal{M}$ with $\geq n$ elements.

It is reasonably clear that if $\mathcal{M} \equiv \mathcal{N}$ then $f_{\mathcal{M}} = f_{\mathcal{N}}$ and $g_{\mathcal{M}} = g_{\mathcal{N}}$ just from looking at sentences.

Let's show that if $f_{\mathcal{M}} = f_{\mathcal{N}}$ and $g_{\mathcal{M}} = g_{\mathcal{N}}$ then $\mathcal{M} \equiv \mathcal{N}$.

We do this with EF-games: the only atomic formula here is $E(x,y)$ so we might as well assume we are playing a game of length n and at the end we will check $E(x_i, x_j)$ for all $i, j \leq n$. Now given n, we define the finite and infinite parts of $\mathcal{M}$ and $\mathcal{N}$:

$$M_{fin} = \{ m \in M : m \text{ lies in an equiv. class with } \leq n \text{ elements} \}$$

$$M_{inf} = M - M_{fin} \text{ and similarly for } N.$$

Now here is the strategy for Player II given that $f_{\mathcal{M}} = f_{\mathcal{N}}$ and $g_{\mathcal{M}} = g_{\mathcal{N}}$.

If Player I plays in M_{fin} (or N_{fin}) then so does Player II. In fact, if Player I plays in an equiv. class with $k \leq n$ elements then so does Player II and if Player I plays in a class played before then Player II plays in the corresponding class in the other model. All of this can be achieved since $f_M = f_N$.

If Player II plays in M_{inf} (or N_{inf}) then so does Player II. If Player I plays in a class previously played, since all the classes have $\geq n$ elements in M_{inf} there is still room for Player II in N_{inf}. If Player I plays in a new class then since $g_M = g_N$ there is also a new class for Player II to pick.

So Player II can win the game of length n. The facts that the values of f_M and g_M are first order expressible gives us all the completions.