Solutions to Assignment #3

1. Fix a ctble model M in a ctble language L. We will build a chain of ctble models M_n s.t. $M_0 = M$, $M_n \prec M_{n+1}$. In order to guarantee that the resulting union is homogeneous we note that for any given M_n there are ctbly many obligations of the following kind: $\bar{a}, \bar{b}, c \in M_n$ such that $\bar{a}$ and $\bar{b}$ have the same type and we are looking for d s.t. $\bar{a}c \equiv \bar{b}d$. d may not occur in M_n but we can find such in some elementary extension. So at each stage we take on ctbly many obligations and the chain is to be ctbly long so we can fulfil them all; that is, we can arrange that if $M^* = \bigcup M_n$ and $\bar{a}, \bar{b} \in M^*$ with $\bar{a} \equiv \bar{b}$ and $c \in M^*$ then $\bar{a}, \bar{b}, c \in M_n$ for some n and hence for some $m > n$ there is $d \in M_m$ with $\bar{a}c \equiv \bar{b}d$ and so M^* is homogeneous.

2. Fix a type p over ctbly many parameters A in

$M^* = \prod_{n \in \mathbb{N}} M_n / \mathcal{U}$. By naming A as ctbly many constants

we can assume $A = \emptyset$ (this just eases notation.) So p is equivalent to a ctble collection of formulas $\{ \varphi_n(x) : n \in \mathbb{N} \}$ s.t. $T \cup \varphi_{n+1}(x) \models \varphi_n(x)$ for all $n \in \mathbb{N}$ where $T = Th(M^*)$.

Now choose a sequence of decreasing sets U_n, $n \in \mathbb{N}$ s.t. $U_n \in \mathcal{U}$ for all n, $\bigcap_{n \in \omega} U_n = \emptyset$ with the property that

$$U_n \subseteq \{ k \in \mathbb{N} : M_k \models \exists x\, \varphi_n(x) \}.$$

Since $\mathcal{U}$ is non-principal, we can guarantee that $\bigcap U_n = \phi$ by letting $U_n = \{k \geq n : M_k \models \exists x\, \varphi_n(x)\}$.

Now let's define a realization of p:
Let $a_k \in \varphi_n(M_k)$ if $k \in U_n$ where n is the maximal i s.t. $k \in U_i$.
(there is such since $\bigcap U_n = \phi$).

Let $\bar{a} \in M^*$ be defined as the class of $\langle a_k : k \in \mathbb{N} \rangle$.

Now for φ_n, $M^* \models \varphi_n(\bar{a})$ since

$\{k \in \mathbb{N} : M_k \models \varphi_n(a_k)\} \supseteq U_n \in \mathcal{U}$

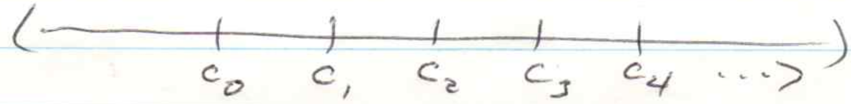
So $\bar{a}$ satisfies φ_n for all n i.e. $\bar{a}$ satisfies p.

3. T is the theory of dense linear orders w/o endpts together with ctbly many constants c_i and axioms

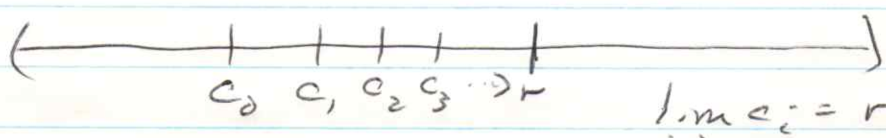
$c_i < c_j$ whenever $i < j$.

Since DLO w/o endpts has quant. elim., the only sentences involving the constants would be Boolean combinations of $c_i < c_j$ and $c_i = c_j$ which are all decided by the given axioms. So T is a complete theory. Here are 3 ctble models of T.

① $\mathbb{Q}$ with the usual ordering and constants c_i forming an unbdd sequences.

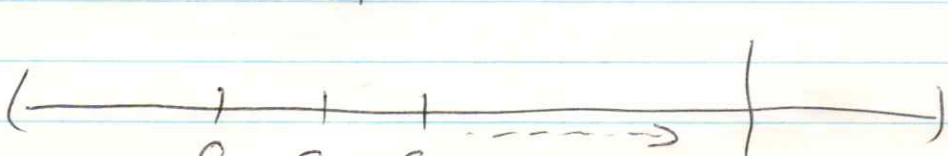


② $\mathbb{Q}$ together with a bdd sequence of c_i's with a limit in $\mathbb{Q}$.



$\lim\limits_{i \to \infty} c_i = r$.

③ $\mathbb{Q}$ together with a bdd sequence of c_i's with no limit in $\mathbb{Q}$.



$\lim\limits_{i \to \infty} c_i$ not rational.

Now take any ctble model of T. We may assume that the underlying set is $\mathbb{Q}$. If the constants are unbdd then this model is easily seen to be isomorphic to (1).

If the constants have a limit in $\mathbb{Q}$ then we get something isomorphic to (2). Otherwise we have something isomorphic to (3).

4. If $A \subseteq B, C$ are all graphs then we can form ~~the~~ an amalgamation of B and C over A as follows: assume the universes of B and C satisfy $B \cap C = A$. Then an amalgamation of B and C over A is ~~the~~ any graph with universe $B \cup C$ and ~~atrtless~~ on B, only edges from B and the same for C. The only freedom here is between vertices in $B-A$ and $C-A$. If $A = \emptyset$ we still count this as an amalgamation.

Now let's build the desired graph. We build it as a union of chains - Let G_0 be any finite graph. If we have built G_n then there are finitely many subgraphs H (including $\emptyset$) and for each such, ctbly many finite extensions. Take on the obligation of, for any pair (H, H') where $H \subseteq G_n$, $H \subseteq H'$, at some future stage, creating an amalgamation of G_m and H' as G_{m+1}.

If $G = \bigcup_{n \in \mathbb{N}} G_n$ then since we allowed $H = \phi$, G contains an isomorphic copy of every finite graph. Moreover, if $H \subseteq G$ ~~and~~, H is finite and $H \subseteq H'$ also finite then by construction there is an isomorphic copy of H' over H contained in G.

Suppose that H is a finite graph with vertices $v_1 \cdots v_n$. Form the formula $\varphi_H(x_1 \cdots x_n) = \bigwedge_{1 \le i < j \le n} R(x_i, x_j)^{\text{if } v_i E v_j}$

where the exponent means the formula is $R(x_i, x_j)$ if there is an edge between v_i and v_j and $\neg R(x_i, x_j)$ otherwise.

Then G satisfies, for every H, $\exists \bar{x}\, \varphi_H(\bar{x})$ and for $H \subseteq H'$ with H' enumerated as $v_1 \cdots v_m$ where $v_1 \cdots v_n$ corresponds to H we have G satisfies.

$\forall \bar{x}\, \varphi_H(\bar{x}) \to \exists \bar{y}\, \varphi_{H'}(\bar{x}, \bar{y})$ ⌜In fact, we could get away with only single point extensions H'⌟

Any other G' which is ctble and satisfies these axioms is isomorphic to G by a routine back-and-forth argument.