Putnam Problems: Pigeonhole Principle, Wednesday February 28

1. What is the largest cardinality of a set S of natural numbers between 2 and 120 such that each pair of numbers in S are coprime, but no element of S is coprime?

2. For any set of twenty distinct numbers from the arithmetic progression $1, 4, 7, \ldots, 100$, prove that two of them add up to 104.

3. Given a set of $n + 1$ integers between 1 and $2n$, prove that one number must divide another. Prove this is not necessarily true for a set of n integers between 1 and $2n$.

4. Given five lattice points $\{P_1, P_2, \ldots, P_5\}$ in the plane, prove that there exists two distinct points P_i, P_j in this set such that the midpoint of $\overline{P_i P_j}$ is also a lattice point. Give an example to show that this fails with only four lattice points.

5. Given five points inside an equilateral triangle of side-length 1, prove that two of the points whose distance apart is at most $\frac{1}{2}$.

6. Show that given any 9 natural numbers it is possible to choose 5 whose sum is divisible by 5.