Mathematical Models: A Catalog of Essential Functions

Section 1.2

Mathematical Models

A **mathematical model** is a description of a real-world phenomenon (for example, climate change, the spread of a virus) using mathematical concepts and language (such as functions and equations).

When we have a model, we can apply tools of *calculus* to study how a quantity *changes*.

Linear Models

slope:

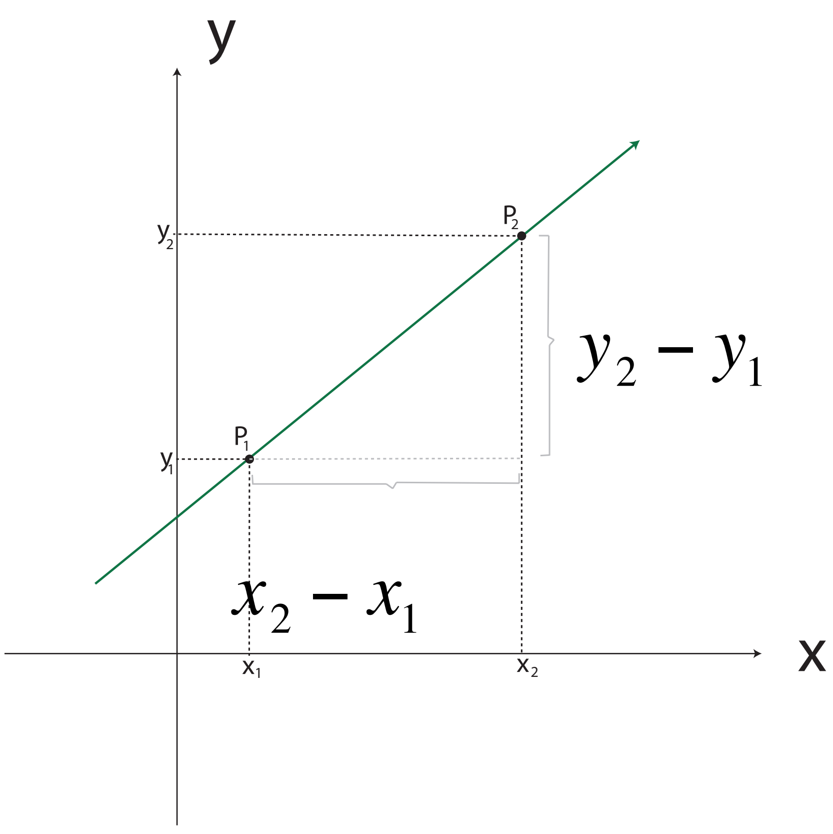
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

point-slope equation:

$$y - y_1 = m(x - x_1)$$

slope-y-intercept equation:

$$y = mx + b$$



Exercise

The relationship between degrees Celsius (C) and degrees Fahrenheit (F) is linear.

We know that $0^o C$ corresponds to $32^o F$ and $100^o C$ corresponds to $212^o F$.

Find the function that converts $^o C$ to $^o F$.

Polynomials

A polynomial is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer (0, 1, 2, 3, 4, …) and the numbers $a_n, a_{n-1}, \cdots, a_1, a_0$ are constants called the coefficients of the polynomial.

Domain:

Degree:

Polynomials

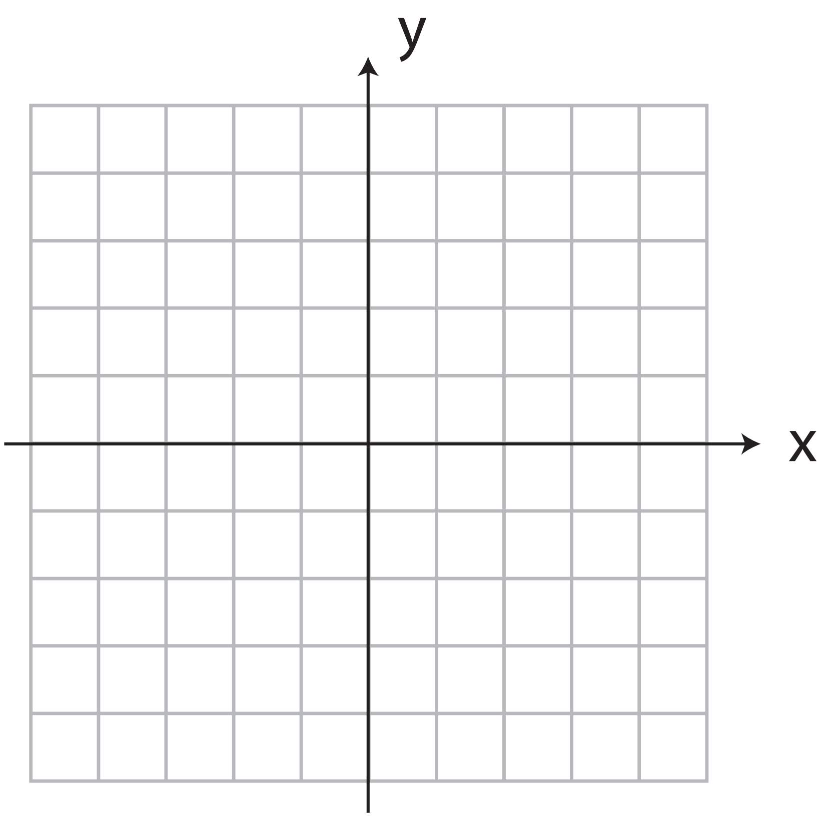
Example:
<u>Quadratic Function:</u>

$$f(x) = -x^2 + 6x - 5$$

Complete the square to find the vertex:

$$f(x) = -(x-3)^2 + 4$$

Domain:
Range:



Polynomials

Note 1:

Polynomials have nice properties (domain is all real numbers, graphs are smooth and continuous, + more…) and for this reason are used in calculus whenever possible for simple calculations

Note 2:

A linear function, $f(x)=mx+b$, is just a polynomial of degree 1.

Power Functions

A power function is a function of the form

$$f(x) = x^a$$

where a is a constant.

Note:

Although a can be any real number, we usually omit the case when $a = 0$.

Power Functions

Some special cases:

a=2: $f(x) = x^2$

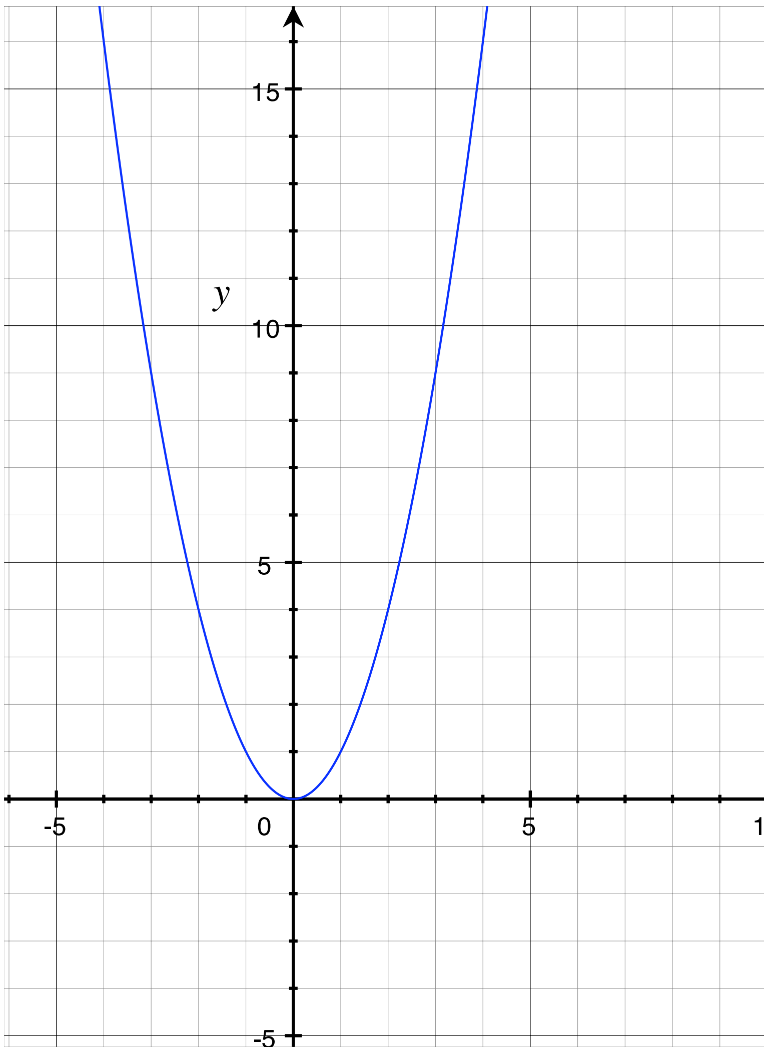
Shape: parabola

Vertex: (0,0)

Domain:

Range:

** Graphs of $x^4, x^6, \ldots$ look similar



Power Functions

Some special cases:

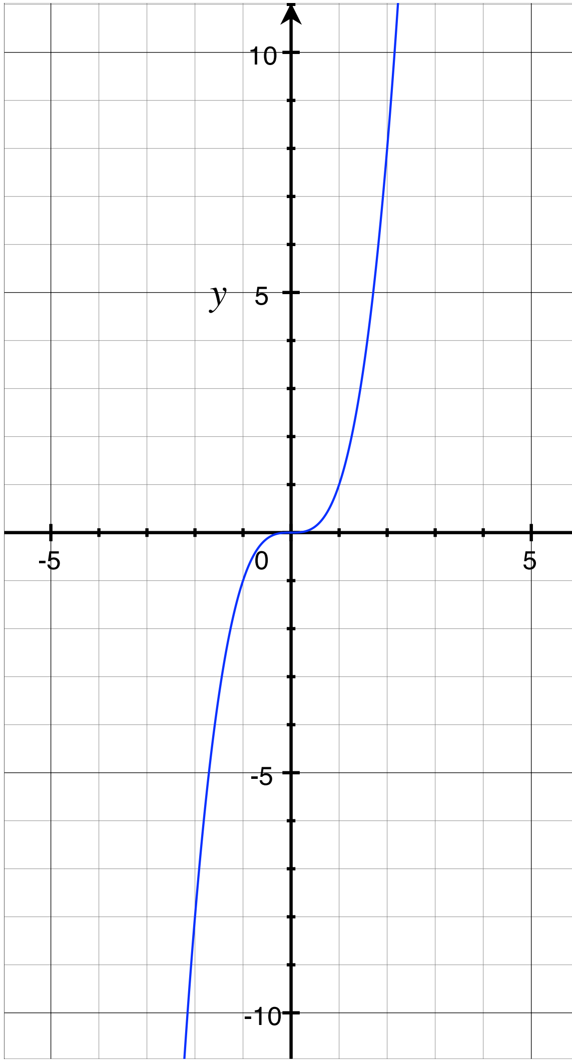
a=3: $f(x) = x^3$

Shape: cubic parabola

Domain:

Range:

**Graphs of $x^5, x^7, \ldots$ look similar



Power Functions

Some special cases:

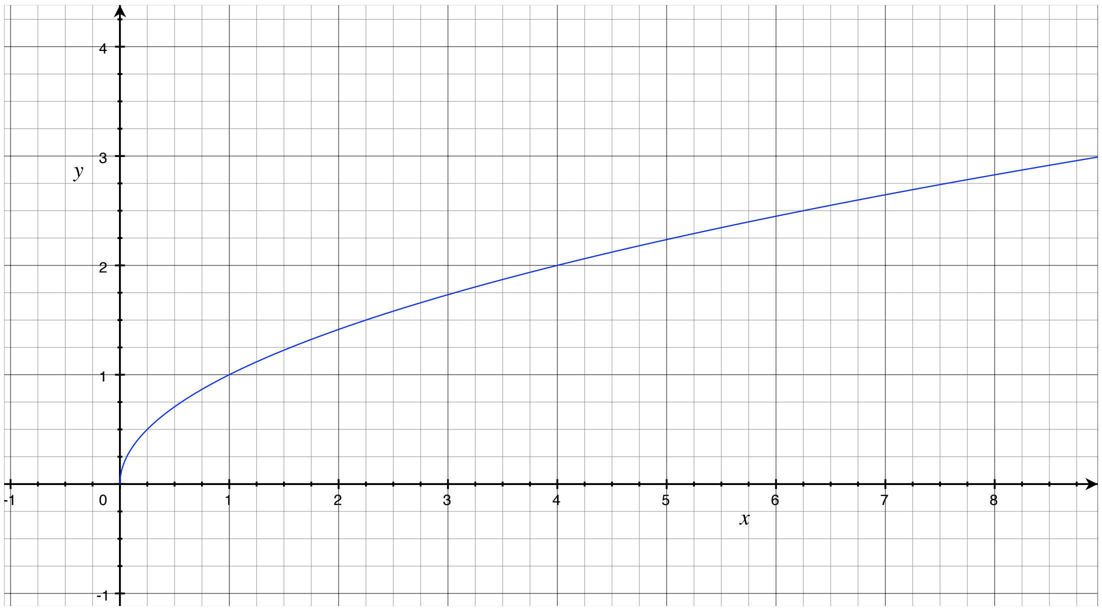
square root function

a=1/2: $f(x) = x^{\frac{1}{2}} = \sqrt{x}$

Shape: half of a parabola

Domain:

Range:



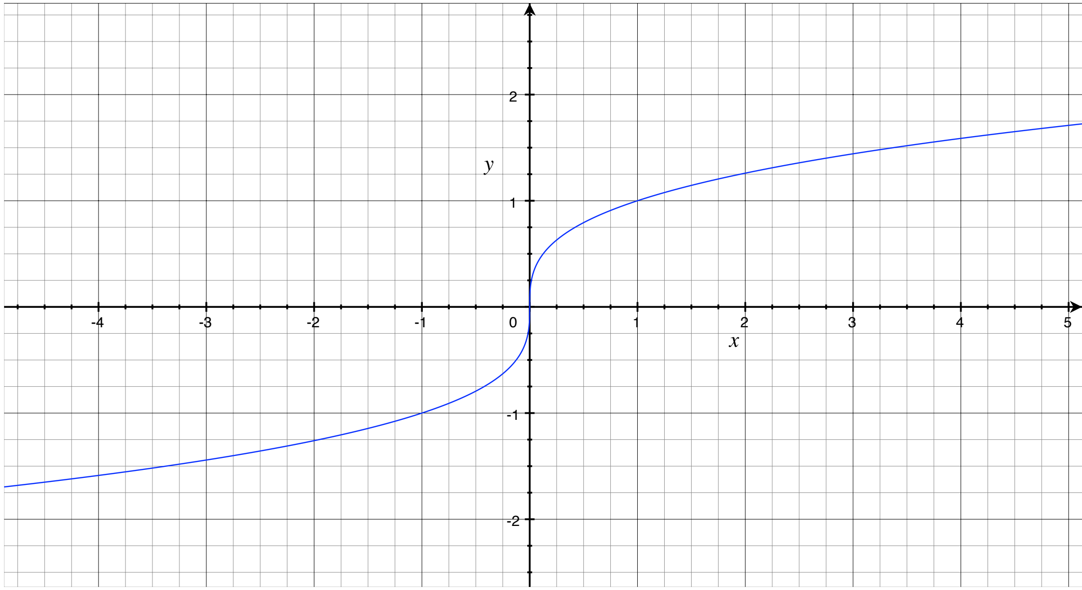
Power Functions

cube root function

Some special cases:

a=1/3: $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$

Shape: cubic parabola



Domain:

Range:

Power Functions

Some special cases:

rational function

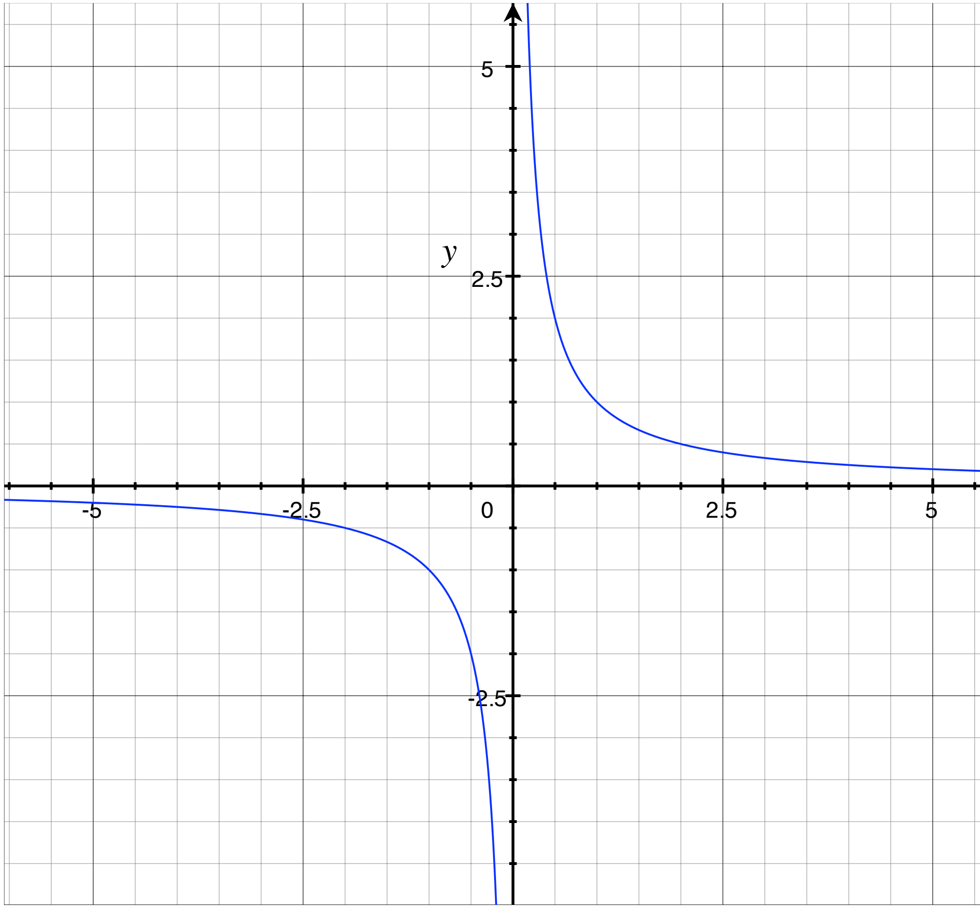
a=-1: $f(x) = x^{-1} = \frac{1}{x}$

Shape: hyperbola

Domain:

Asymptotes:

Range:

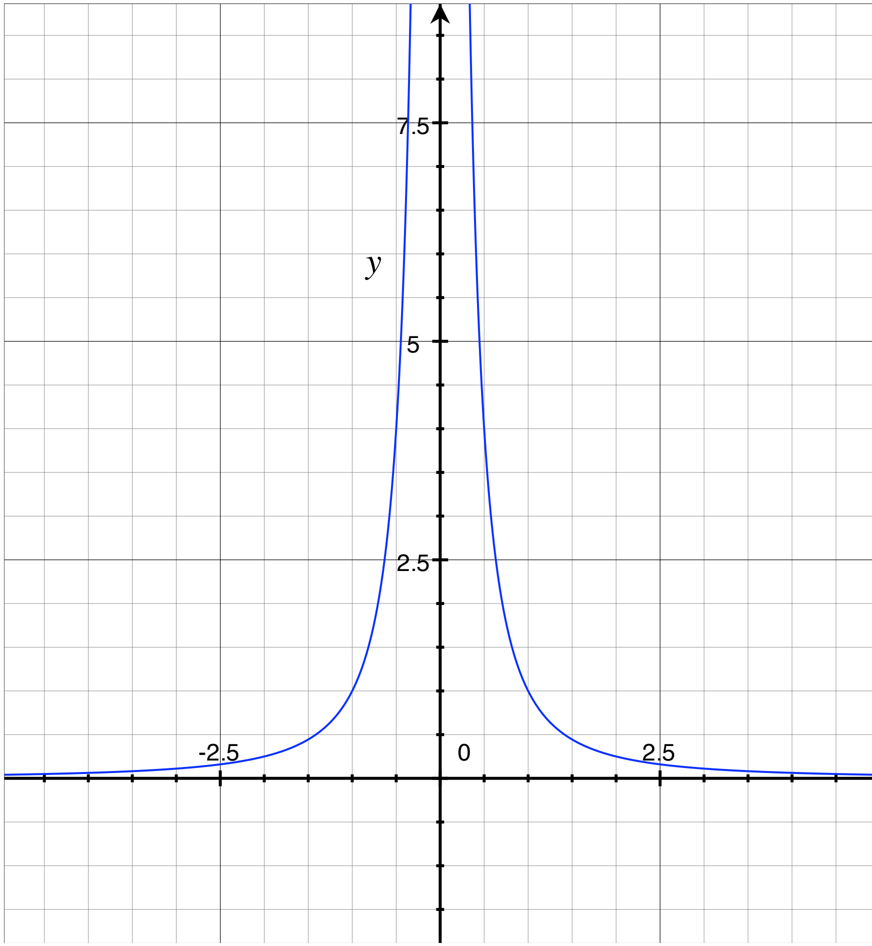


Power Functions

Some special cases:

rational function

a=-2: $f(x) = x^{-2} = \dfrac{1}{x^2}$



Shape: hyperbola

Domain:

Asymptotes:

Range:

Exercise

Example: Blood Circulation Time in Mammals

Blood circulation time is the average time needed for the blood to reach a site in the body and come back to the heart. It has been determined that, for mammals, the blood circulation time is *proportional* to the fourth root of the body mass.

Find a formula to model this relationship and sketch its graph.

Rational Functions

A rational function *f* is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials and $Q(x) \neq 0.$

Examples:

$$f(x) = \frac{1}{x-3} \qquad f(x) = \frac{x+2}{x^2-4}$$

Exercise

Sketch the graphs of the rational functions:

(a) $f(x) = \dfrac{x+2}{x^2-4}$

(b) $g(x) = \dfrac{x^2-4}{x+2}$

Algebraic Functions

A combination of any of the the previous functions using algebraic operations $(+, -, \times, \div, \sqrt[n]{\ })$ is called an *algebraic function*.

Example: $f(x) = \dfrac{3x - 4}{\sqrt{x^2 + 1}}$

Trigonometric Models

Trigonometric functions are periodic functions that are used to model quantities that oscillate.

Note: In calculus, the convention is that r**adian measure** is always used, unless explicitly stated otherwise.

Exercise

Example: Seasonal Growth

A population of river sharks in New Zealand changes periodically with a period of 12 months. In January, the population reaches a maximum of 14, 000, and in July, it reaches a minimum of 6, 000.

Using a trigonometric function, find a formula which describes how the population of river sharks changes with time.