Inverse Functions and Logarithms

Section 1.5

One-To-One Functions

A function *f* is called **one-to-one** if it never takes on the same value twice; that is

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2$$

Note: A function has an **inverse** if and only if it is a one-to-one function.

Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Exercise

(a) Give an example of a function that is one-to-one. Draw its graph.

(b) Sketch the graph of $f(x) = \sin x$. Determine the largest interval around the origin on which f is one-to-one.

Inverse Functions

Definition:

Let *f* be a one-to-one function with domain *A* and range *B*. Then its inverse function f^{-1} has domain *B* and range *A* and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any *y* in *B*.

Cancellation equations:

$f^{-1}(f(x)) = x$ for every x in A

$f(f^{-1}(x)) = x$ for every x in B

Finding the Inverse of a Function

<u>Algorithm:</u>

1. Write the equation y=f(x).
2. Solve for x in terms of y.
3. Replace x by f^{-1} (x) and y by x.

<u>*Note*</u>: domain of f^{-1} = range of *f*
range of f^{-1} = domain of *f*

Exercise

Find the inverse of the following functions. State domain and range.

(a) $f(x) = \dfrac{2x - 1}{x + 3}$

(b) $T(B) = a\sqrt[4]{B}$

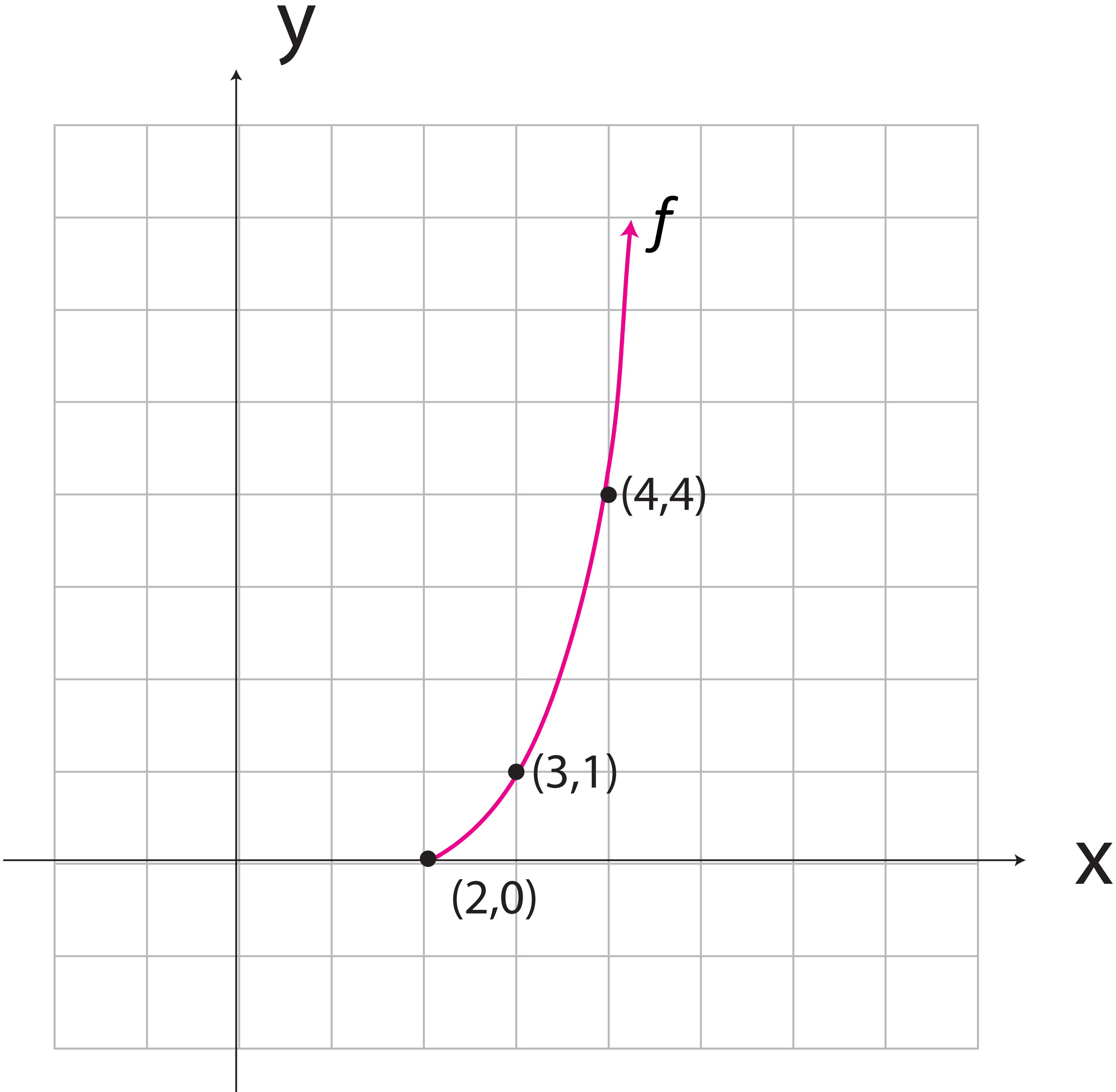
Note: Since the variables *T* and *B* have a physical meaning, we do not interchange them

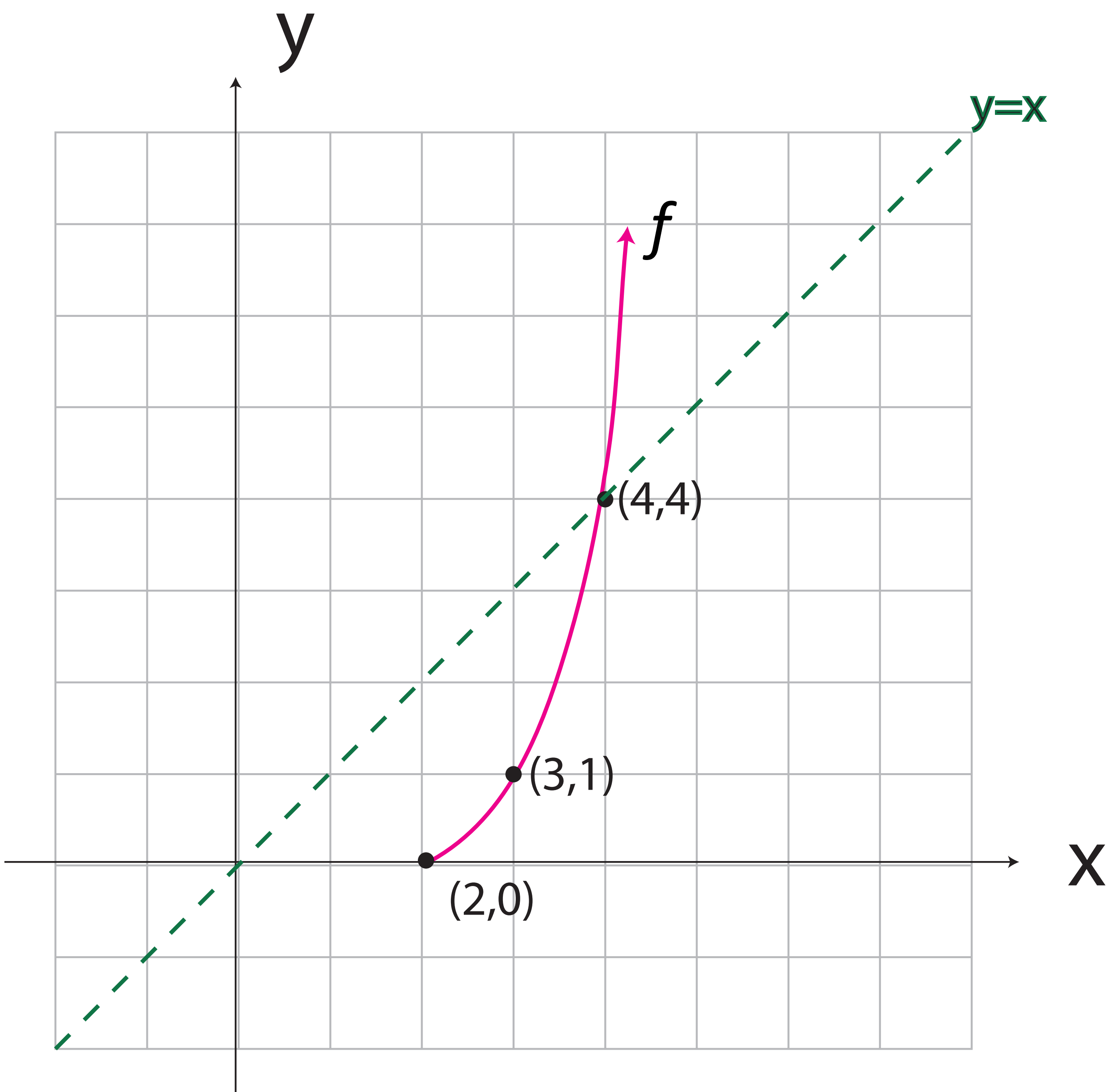
Graphs of f^{-1} and f

The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

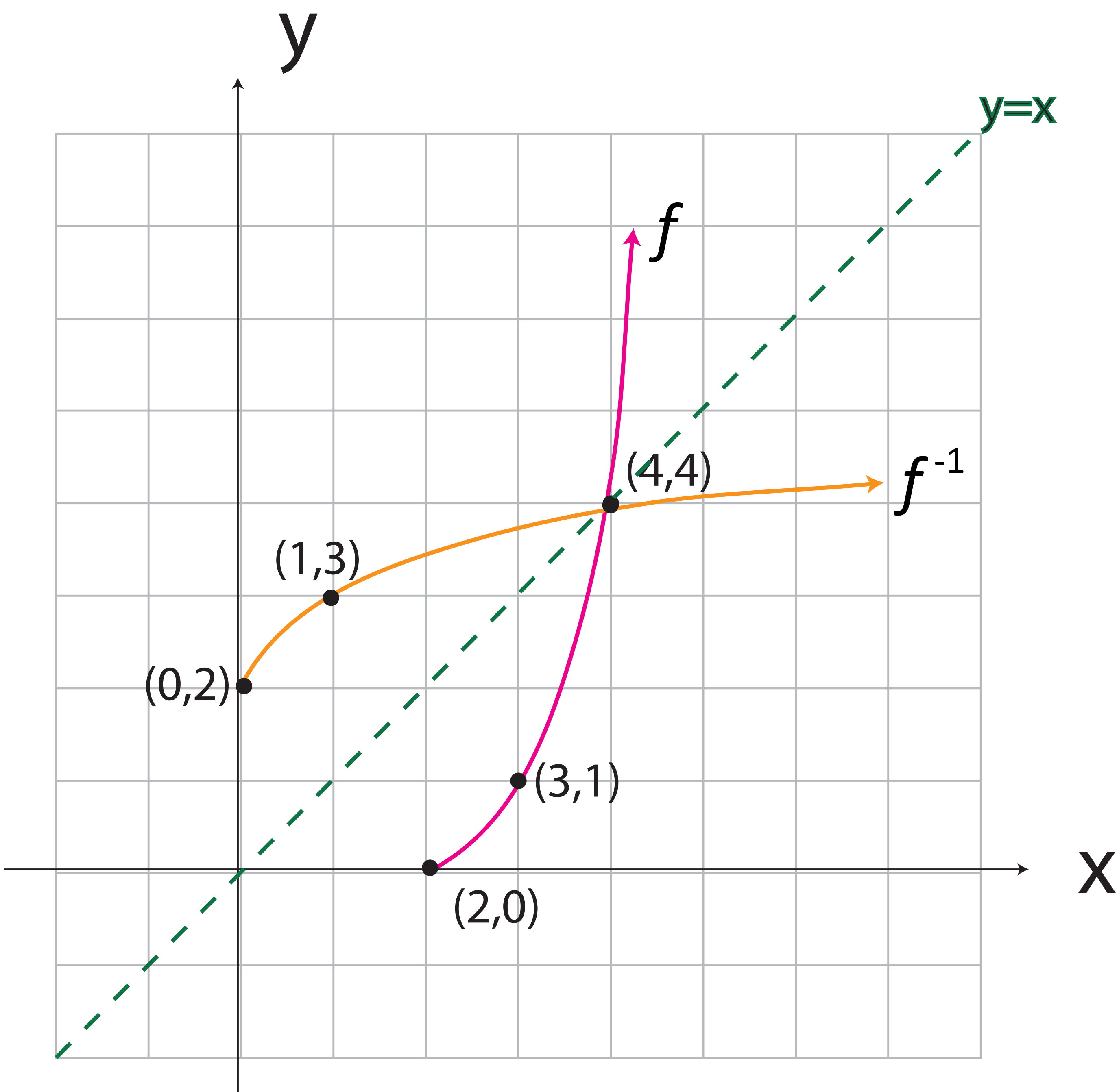
Points (x,y) on f become the points (y,x) on f^{-1}.

Example:
Given $f(x) = (x-2)^2, x \geq 2$ sketch f and f^{-1}.


y
f
(4,4)
(3,1)
x
(2,0)



section 1.5



y
y=x
f
f^{-1}
(4,4)
(1,3)
(0,2)
(3,1)
(2,0)
x

Logarithmic Functions

The inverse of an exponential function is a logarithmic function, i.e.

$$\text{If } f(x) = a^x, \text{ then } f^{-1}(x) = \log_a x.$$

Cancellation equations:

In general:

$$f(f^{-1}(x)) = x$$

$$f^{-1}(f(x)) = x$$

For exponentials & logarithms:

$$a^{\log_a x} = x$$

$$\log_a a^x = x$$

$$e^{\ln x} = x$$

$$\ln e^x = x$$

Graphs of Logarithmic Functions

Recall:

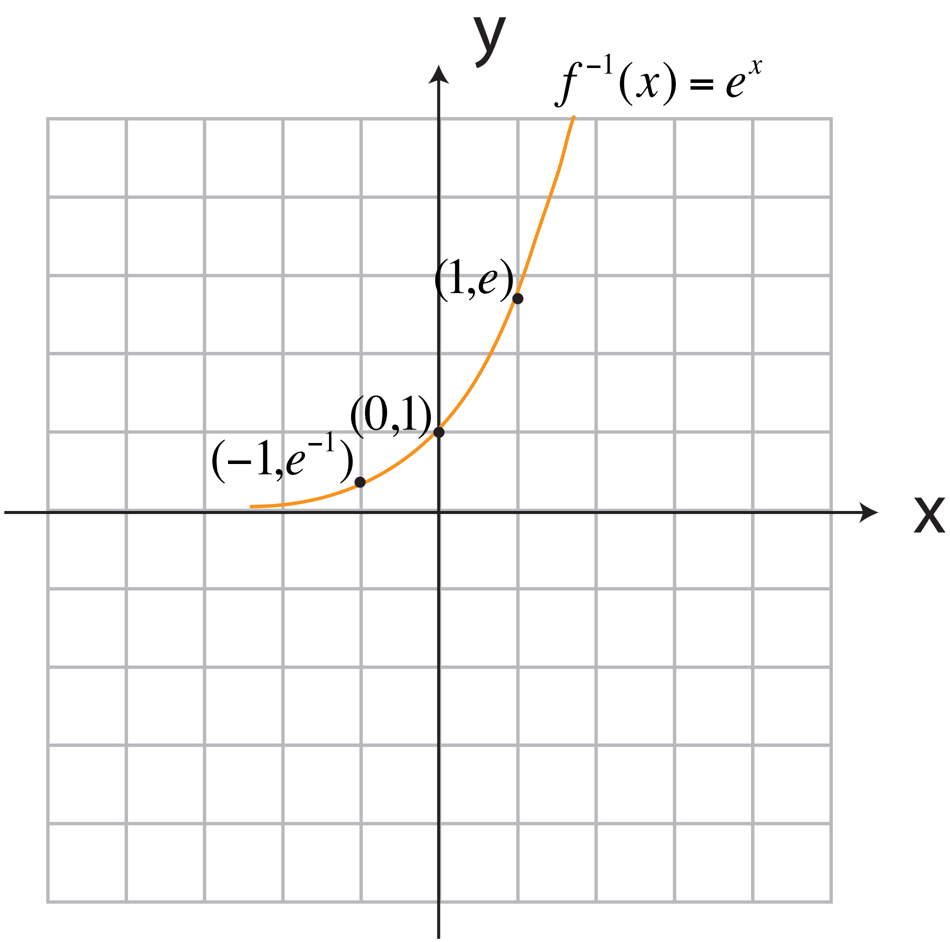
For inverse functions, the domain and range are interchanged and their graphs are reflections in the line $y = x$.

Example:

Graph $f(x) = \ln x$.

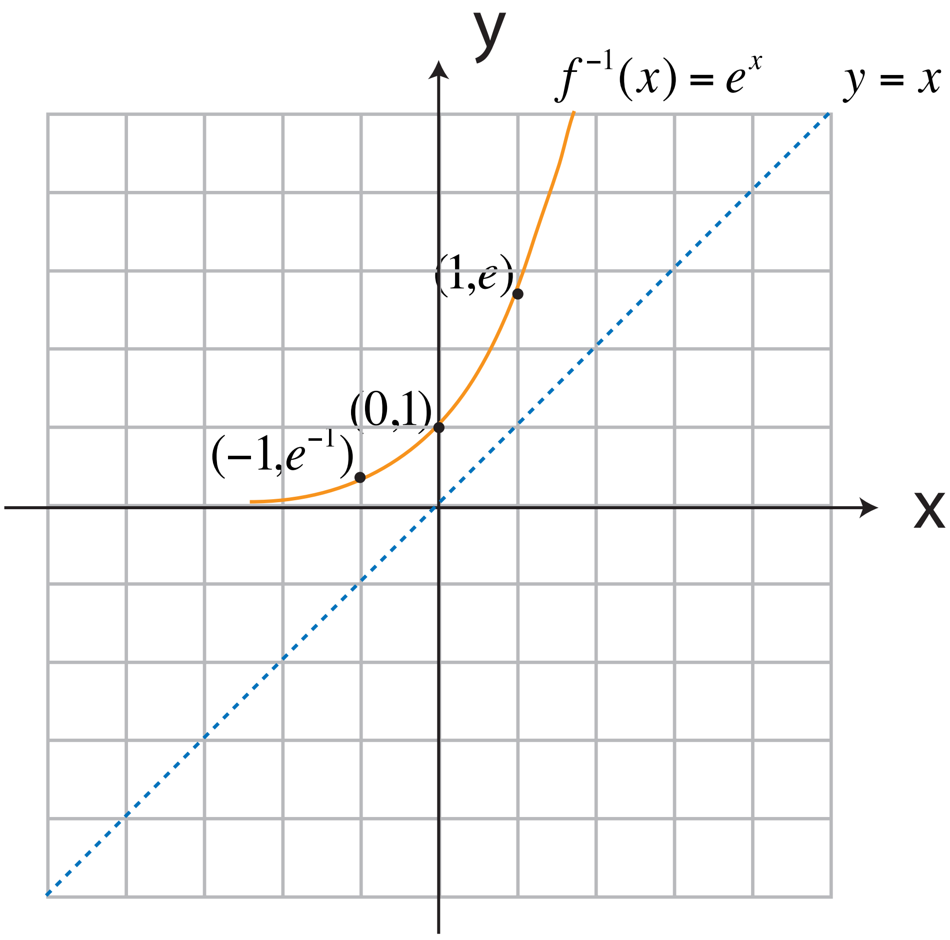
Graphs of Logarithmic Functions

$e \approx 2.7$



Graphs of Logarithmic Functions

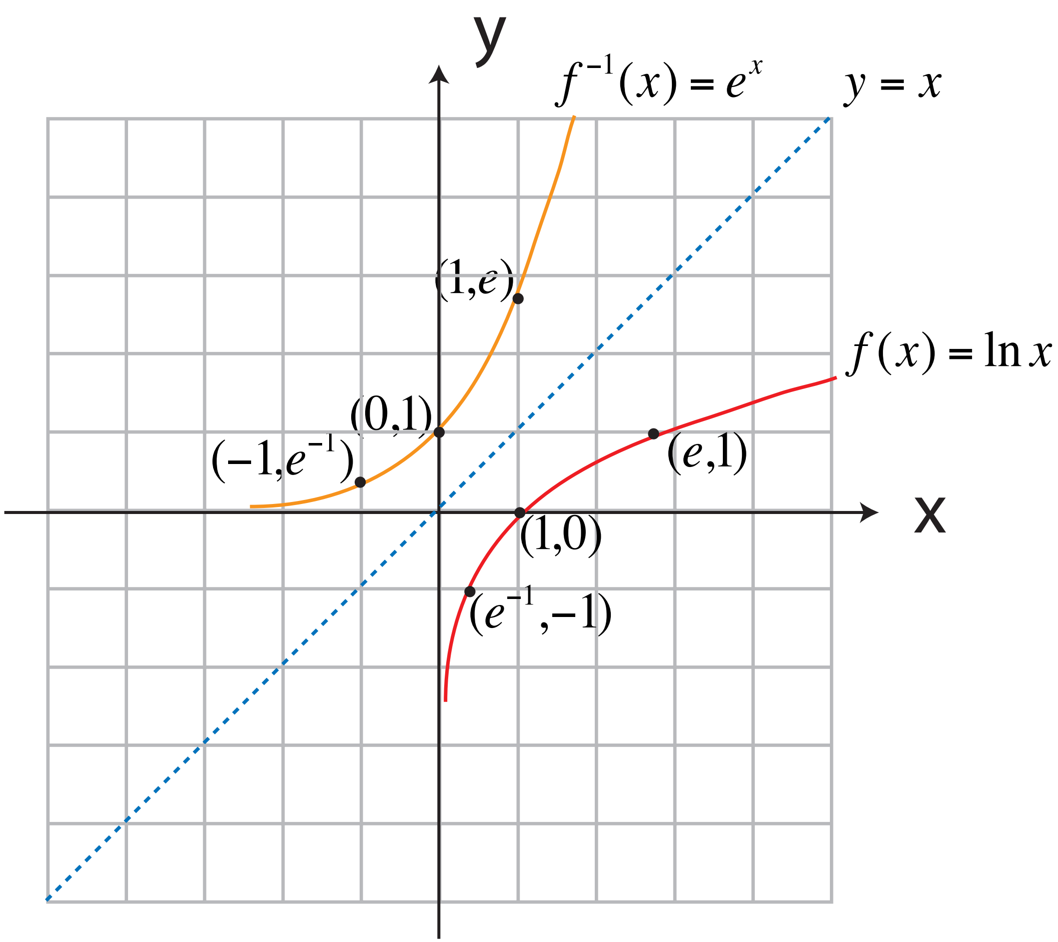
$e \approx 2.7$



Graphs of Logarithmic Functions

$e \approx 2.7$

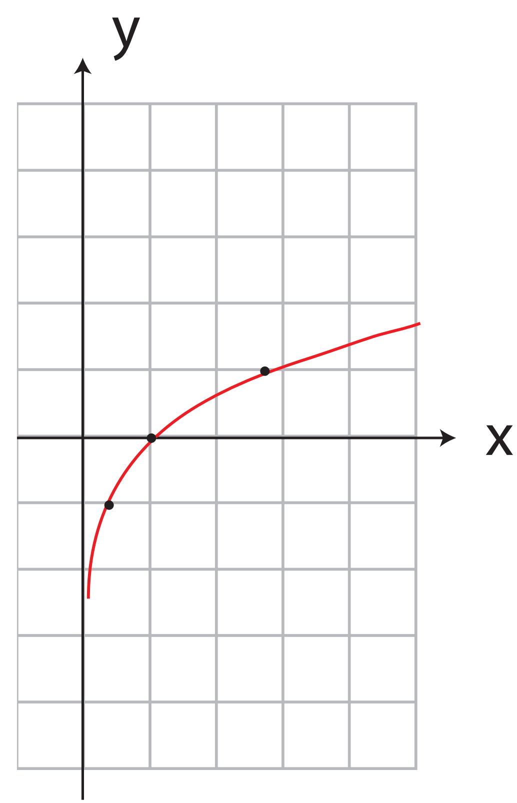
Memorize!!!



The Natural Logarithm

$$f(x) = \ln x$$

Domain: $x > 0$
Range: $y \in R$



Graph:
The graph increases from negative infinity near x=0 (vertical asymptote) and rises more and more slowly as x becomes larger.

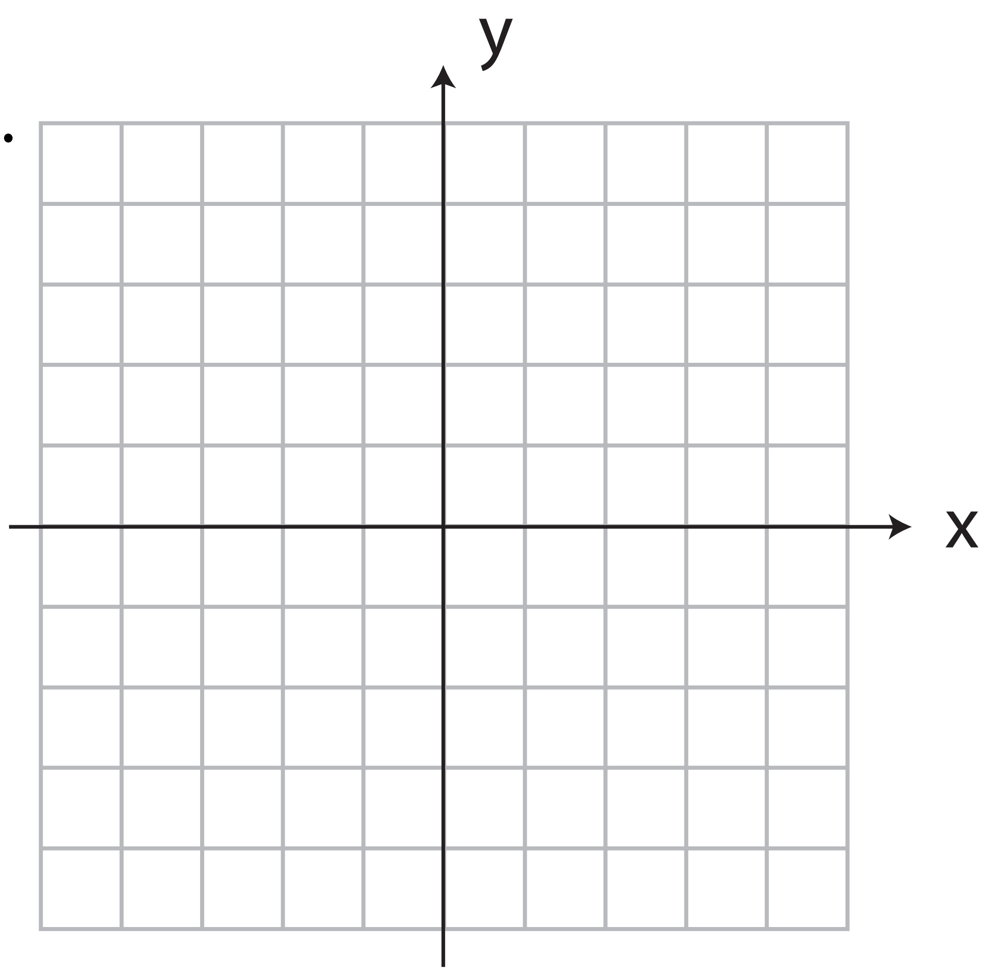
Note:

$\ln e = 1$ and $\ln 1 = 0$

Example

Graph $f(x) = \log_3(2 - x)$.

State the domain.



Laws of Logs

For x,y>0 and p any real number:

$$\ln(xy) = \ln x + \ln y$$

$$\ln(x/y) = \ln x - \ln y$$

$$\ln(x^p) = p \ln x$$

Example:

Simplify, if possible.

(a) $\ln\left(\frac{1}{e}\right)$ (b) $\log_2 4 + \log_2 16 - \log_2 2$

(c) $\log_5(5 \cdot 8)$ (d) $\ln(1+e)$

Exercises

Solve the following equations for x.

(a) $e^{x+1} = 8$

(b) $2 \cdot 3^x = 8$

(c) $\log_2(x^2 - 1) = 3$

(d) $\ln x + \ln(x-1) = \frac{1}{2}\ln 36$

Exponential Models

When the change in a measurement is proportional to its size, we can describe the measurement as a function of time by the formula

$$S(t) = S(0)e^{\alpha t}$$

where

$S(t)$ is the value of the measurement at time t

$S(0)$ is the initial value of the measurement, and

α is a parameter which describes the rate at which the measurement changes

Exercise

A bacterial culture starts with 100 bacteria and after 3 hours the population is 450 bacteria.

Assuming that the rate of growth of the population is proportional to its size, find the time it takes for the population to double.