

Limits at Infinity; Horizontal Asymptotes

Section 2.6

Limits “at” Infinity

The behaviour of functions at infinity is also known as “end behaviour”.

What happens to the y -values of a function $f(x)$ as the x -values increase or decrease without bounds?

$$\lim_{x \rightarrow -\infty} f(x) = ?$$

$$\lim_{x \rightarrow \infty} f(x) = ?$$

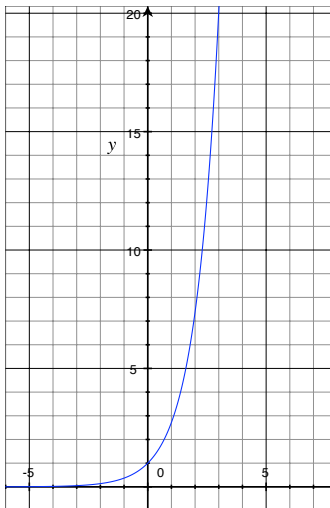
Limits at Infinity

Possibility:

y-values also approach infinity or negative infinity

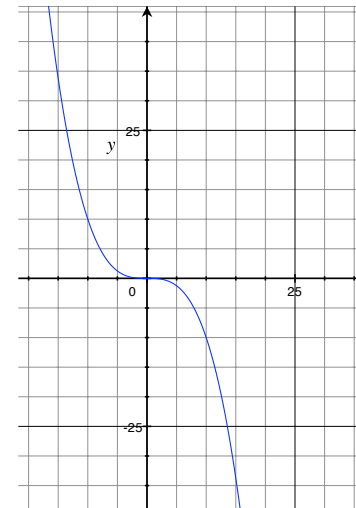
Examples:

$$f(x) = e^x$$



$$\lim_{x \rightarrow \infty} e^x = \infty \quad (\text{limit D.N.E})$$

$$f(x) = -0.01x^3$$



$$\lim_{x \rightarrow \infty} -0.01x^3 = -\infty \quad (\text{limit D.N.E})$$

Limits at Infinity

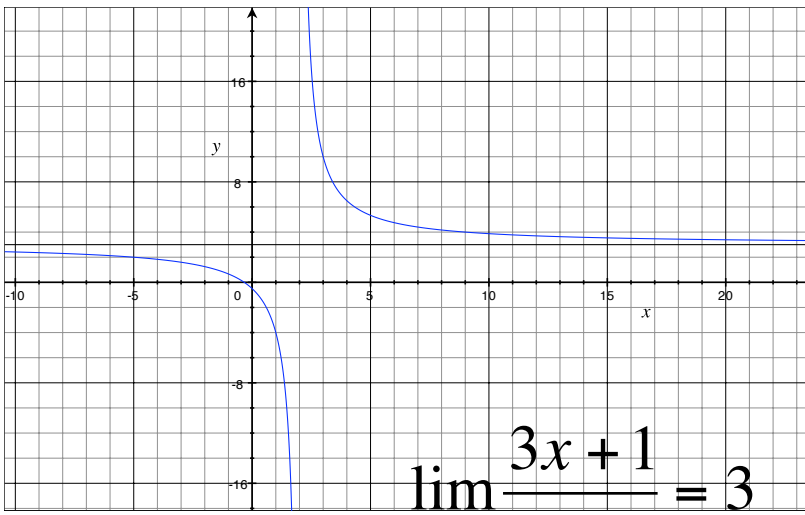
Possibility:

y-values approach a unique real number L

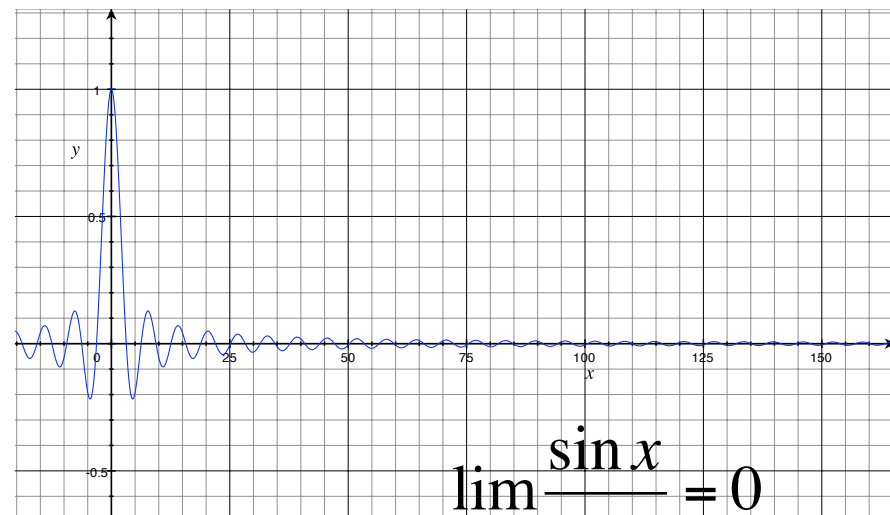
Examples:

$$f(x) = \frac{3x + 1}{x - 2}$$

$$f(x) = \frac{\sin x}{x}$$



$$\lim_{x \rightarrow \infty} \frac{3x + 1}{x - 2} = 3$$



$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

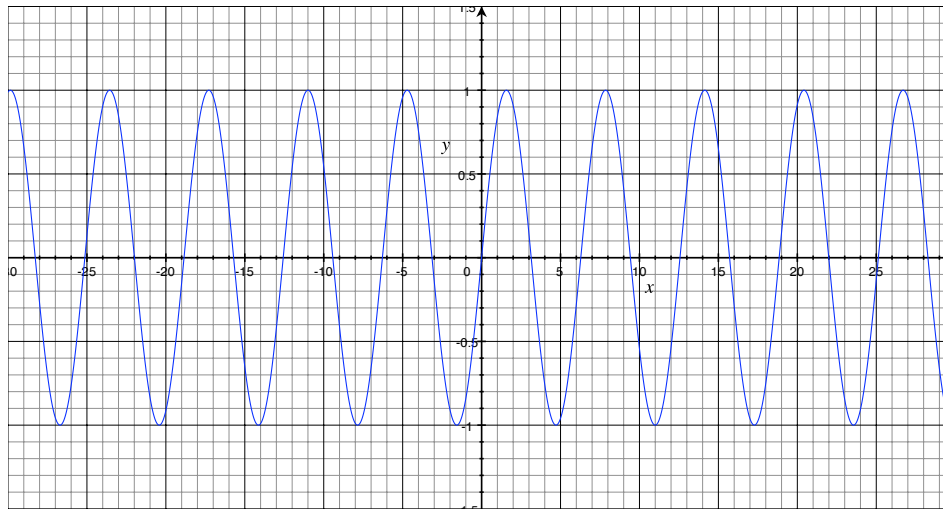
Limits at Infinity

Possibility:

y -values oscillate and do not approach a single value

Example:

$$f(x) = \sin x$$



$$\lim_{x \rightarrow \infty} \sin x \text{ D.N.E.}$$

Limits at Infinity

Definition:

$$\lim_{x \rightarrow \infty} f(x) = L$$

“the limit of $f(x)$, as x approaches ∞ , equals L ”

means that the values of $f(x)$ (y -values) can be made as close as we'd like to L by taking x sufficiently large.

Definition:

$$\lim_{x \rightarrow -\infty} f(x) = L$$

“the limit of $f(x)$, as x approaches $-\infty$, equals L ”

means that the values of $f(x)$ (y -values) can be made as close as we'd like to L by taking x sufficiently small.

Calculating Limits at Infinity

* The Limit Laws listed in 2.3 (except 9 and 10) are still valid if “ $x \rightarrow a$ ” is replaced by “ $x \rightarrow \infty$ ”

Theorem:

If $r > 0$ is a rational number, then $\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$.

If $r > 0$ is a rational number such that x^r is defined for all x , then $\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0$.

Calculating Limits at Infinity

Examples:

Find the limit or show that it does not exist.

$$(a) \lim_{x \rightarrow \infty} \frac{3x - 7}{2\sqrt{x} + 1}$$

$$(b) \lim_{x \rightarrow \infty} \frac{x - 1}{x^2 + 3x + 4}$$

$$(c) \lim_{x \rightarrow -\infty} \left(x + \sqrt{x^2 + 2x} \right)$$

Horizontal Asymptotes

Definition:

The line $y=L$ is called a **horizontal asymptote** of the curve $y=f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L$$

Asymptotes

Example:

Determine the value of A such that $f(x) = \frac{Ax + 1}{3x - 4}$ has horizontal asymptote $y = 1/2$.

End Behaviour

Example:

Determine the end behaviour of $f(x) = 4x^2 - x^5$.