MATH 1F03 - ASSIGNMENT 12

Sections 12.4 and 12.5

1. Find the cross product $\vec{a} \times \vec{b}$ and verify that it is orthogonal to both $\vec{a}$ and $\vec{b}$.

(a) $\vec{a} = <4, 3, -2>$, $\vec{b} = <2, -1, 1>$

$\vec{a} \times \vec{b} = <1, -8, -10>$

$$\begin{array}{cc} \cancel{4} & \cancel{2} \\ 3 & -1 \\ -2 & 1 \\ 4 & 2 \\ 3 & -1 \\ \cancel{-2} & \cancel{1} \end{array}$$

$x = (3)(1) - (-2)(-1) = 1$

$y = (-2)(2) - (4)(1) = -8$

$z = (4)(-1) - (3)(2) = -10$

$\vec{a} \cdot (\vec{a} \times \vec{b}) = (4)(1) + (3)(-8) + (-2)(-10) = 4 - 24 + 20 = 0$
$\Rightarrow \vec{a} \perp \vec{a} \times \vec{b}$

$\vec{b} \cdot (\vec{a} \times \vec{b}) = (2)(1) + (-1)(-8) + (1)(-10) = 0 \Rightarrow \vec{b} \perp \vec{a} \times \vec{b}$

(b) $\vec{a} = 2\mathbf{j} - 4\mathbf{k}$, $\vec{b} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$\vec{a} \times \vec{b} = 14\hat{i} + 4\hat{j} + 2\hat{k}$

$$\begin{array}{cc} \cancel{0} & \cancel{-1} \\ 2 & 3 \\ -4 & 1 \\ 0 & -1 \\ 2 & 3 \\ \cancel{-4} & \cancel{1} \end{array}$$

$x = (2)(1) - (-4)(3) = 14$

$y = (-4)(-1) - (0)(1) = 4$

$z = (0)(3) - (2)(-1) = 2$

$\vec{a} \cdot (\vec{a} \times \vec{b}) = (0)(14) + 2(4) + (-4)(2) = 0 \Rightarrow \vec{a} \perp (\vec{a} \times \vec{b})$

$\vec{b} \cdot (\vec{a} \times \vec{b}) = (-1)(14) + (3)(4) + (1)(2) = 0 \Rightarrow \vec{b} \perp (\vec{a} \times \vec{b})$

2. (a) If $\vec{a} = \mathbf{i} - 2\mathbf{k}$ and $\vec{b} = \mathbf{j} + \mathbf{k}$, find $\vec{a} \times \vec{b}$.

$\vec{a} \times \vec{b} = 2\hat{i} - \hat{j} + \hat{k}$

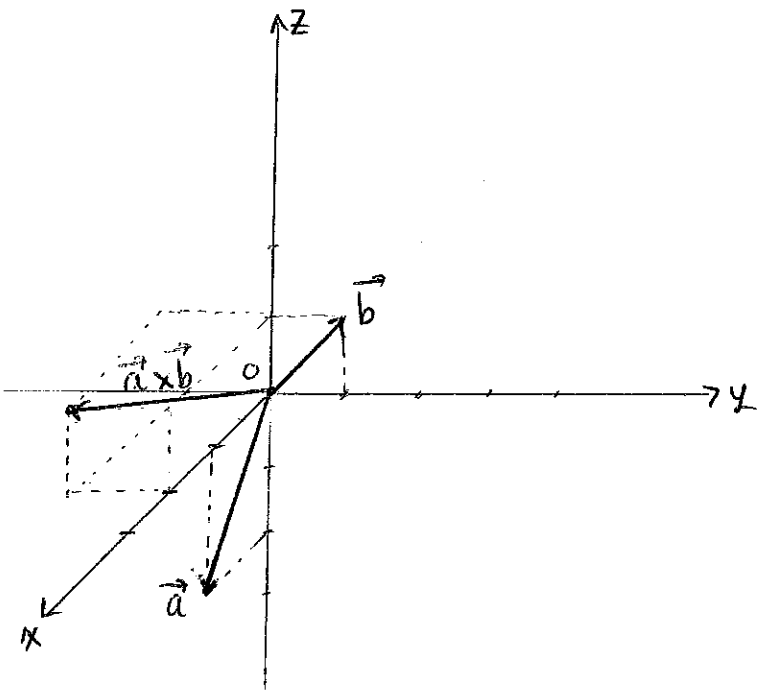
$$\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ -2 & 1 \\ 1 & 0 \\ 0 & 1 \\ -2 & 1 \end{array}$$

$x = (0)(1) - (-2)(1) = 2$

$y = (-2)(0) - (1)(1) = -1$

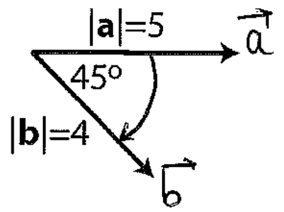
$z = (1)(1) - (0)(0) = 1$

(b) Sketch $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ as vectors starting at the origin.



3. Find $|\vec{a} \times \vec{b}|$ and determine whether $\vec{a} \times \vec{b}$ is directed into the page or out of the page.

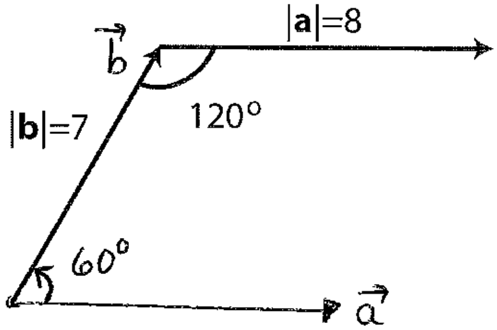
(a)



$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$
$$= (5)(4)\sin 45°$$
$$= \frac{20}{\sqrt{2}}$$

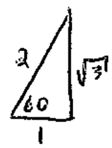
$\vec{a} \times \vec{b}$ points <u>into</u> the page.

(b)



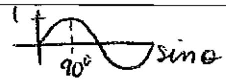
$\vec{a}$ and $\vec{b}$ must be tail-to-tail

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$
$$= (8)(7)\sin 60°$$
$$= 56\left(\frac{\sqrt{3}}{2}\right)$$
$$= 28\sqrt{3}$$



$\vec{a} \times \vec{b}$ points <u>out of</u> the page.

4. question 16 on page 821 in your textbook

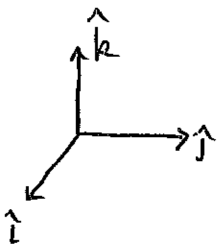


(a) $|\vec{a} \times \vec{b}| = (3)(2)\sin 90^\circ = 6$

(b) $\vec{a} \times \vec{b} = \langle d, e, f \rangle$

$\oplus$ $\ominus$ 0

5. Find the vector using properties of cross products.



(a) $\mathbf{k} \times (\mathbf{i} - 2\mathbf{j})$

$= \hat{k} \times \hat{\imath} - 2(\hat{k} \times \hat{\jmath})$

$= \hat{\jmath} - 2(-\hat{\imath})$

$= 2\hat{\imath} + \hat{\jmath}$

(b) $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$

$= \hat{\imath} \times \hat{\imath} + \hat{\imath} \times (-\hat{\jmath}) + \hat{\jmath} \times \hat{\imath} + \hat{\jmath} \times (-\hat{\jmath})$

$= \vec{0} - (\hat{\imath} \times \hat{\jmath}) + \hat{\jmath} \times \hat{\imath} - (\underbrace{\hat{\jmath} \times \hat{\jmath}}_{=\vec{0}})$

$= -\hat{k} - \hat{k}$

$= -2\hat{k}$

6. Find two unit vectors orthogonal to both $\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$.

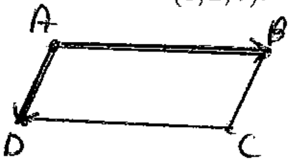
$$(\hat{j}-\hat{k})\times(\hat{i}+\hat{j}) = \underbrace{\hat{j}\times\hat{i}}_{=-\hat{k}} + \underbrace{\hat{j}\times\hat{j}}_{=0} - \underbrace{\hat{k}\times\hat{i}}_{=\hat{j}} - \underbrace{\hat{k}\times\hat{j}}_{=-\hat{i}}$$

$$= \hat{i} - \hat{j} - \hat{k}$$

$$|\hat{i}-\hat{j}-\hat{k}| = \sqrt{1^2+(-1)^2+(-1)^2} = \sqrt{3}$$

$$\therefore \hat{u} = \frac{1}{\sqrt{3}}(\hat{i}-\hat{j}-\hat{k}) = \frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

7. Find the area of the parallelogram with vertices $A(1,0,2)$, $B(3,3,3)$, $C(7,5,8)$, and $D(5,2,7)$.



$$\text{Area} = |\vec{AB} \times \vec{AD}|$$

$$\vec{AB} = \langle 3-1, 3-0, 3-2\rangle = \langle 2,3,1\rangle$$

$$\vec{AD} = \langle 5-1, 2-0, 7-2\rangle = \langle 4,2,5\rangle$$

$$\vec{AB} \times \vec{AD} = \langle 13,-6,-8\rangle$$

$\cancel{2}$ $\cancel{4}$
3 2
1 5
2 4
3 2
$\cancel{1}$ $\cancel{5}$

$x = (3)(5) - (1)(2) = 13$

$y = (1)(4) - (2)(5) = -6$

$z = (2)(2) - (3)(4) = -8$

$$\therefore \text{Area} = \sqrt{13^2+(-6)^2+(-8)^2} = \sqrt{269} \approx 16.4$$

8. Consider the line $y = 1.5x + 2$.

(a) Find a vector equation, parametric equations, and symmetric equations for the line.

$m = \frac{\Delta y}{\Delta x} = 1.5 = \frac{3}{2} \Rightarrow \vec{v} = \langle 2, 3 \rangle$

↑ any vector parallel to this vector would also work

Let $\vec{r}_0 = \langle 0, 2 \rangle$.

vector eq^N: $\vec{r} = \langle 0, 2 \rangle + s\langle 2, 3 \rangle$, $s \in \mathbb{R}$

parametric eq^{N}s: $x = 2s$, $y = 2 + 3s$

symmetric eq^{N}s: $\frac{x}{2} = \frac{y-2}{3}$

(b) Does the line $y = 1.5x + 2$ intersect the line $x = 1 + t, y = 6 - 6t$? If so, find the point of intersection.

set coordinates of each line equal + solve system of eq^{N}s:

$2s = 1 + t$ ①

$2 + 3s = 6 - 6t$ ②

6① + ②: $12s = 6 + 6t$

$2 + 3s = 6 - 6t$

$2 + 15s = 12$

$\Rightarrow \boxed{s = \frac{10}{15} = \frac{2}{3}}$

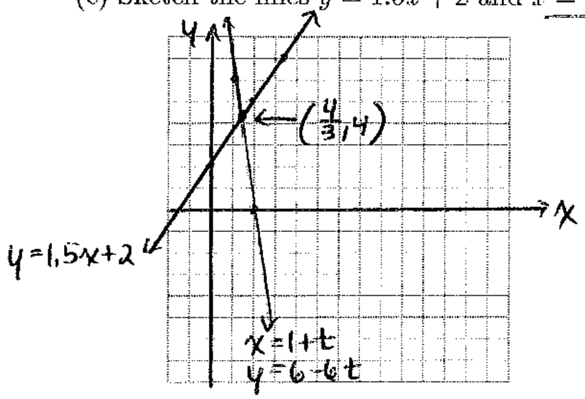
sub $s = \frac{2}{3}$ into ②:

$2 + 3\left(\frac{2}{3}\right) = 6 - 6t \Rightarrow \boxed{t = \frac{1}{3}}$

either sub $s = \frac{2}{3}$ into line in part (a) or $t = \frac{1}{3}$ into line in part (b) to get the point of intersection!

$\left(\frac{4}{3}, 4\right)$

(c) Sketch the lines $y = 1.5x + 2$ and $x = 1 + t, y = 6 - 6t$.



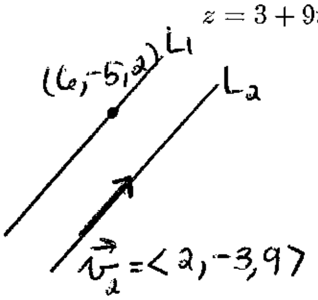
$t = 0 \Rightarrow x = 1, y = 6$

$t = 1 \Rightarrow x = 2, y = 0$

9. Find a vector equation, parametric equations, and symmetric equations for each line.

(a) The line through the point $(6, -5, 2)$ and parallel to the line $x = -1 + 2t, y = 6 - 3t,$ $z = 3 + 9t$

L_2



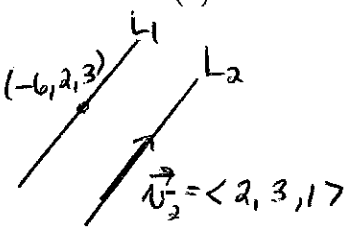
$L_1 \parallel L_2 \Rightarrow v_1 \parallel v_2$

$\therefore L_1: \vec{r} = \langle 6,-5,2 \rangle + s\langle 2,-3,9 \rangle, \; s \in \mathbb{R}$

$x = 6 + 2s, \; y = -5 - 3s, \; z = 2 + 9s$

$$\frac{x-6}{2} = \frac{y+5}{-3} = \frac{z-2}{9}$$

(b) The line through the point $(-6, 2, 3)$ and parallel to the line $\frac{1}{2}x = \frac{1}{3}y = z + 1$

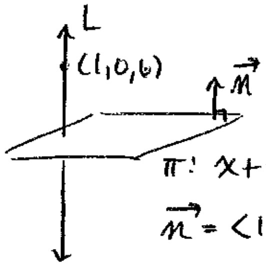


$L_1: \vec{r} = \langle -6,2,3 \rangle + t\langle 2,3,1 \rangle, \; t \in \mathbb{R}$

$x = -6 + 2t, \; y = 2 + 3t, \; z = 3 + t$

$$\frac{x+6}{2} = \frac{y-2}{3} = z - 3$$

(c) The line through the point $(1,0,6)$ and perpendicular to the plane $x+3y+z=5$



π: $x+3y+z=5$

$\vec{n} = \langle 1,3,1 \rangle$

$\vec{n} \parallel \vec{v}$

normal vector for plane

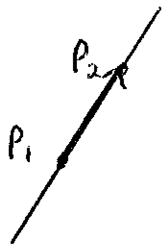
direction vector for line

$\therefore$ L: $\vec{r} = \langle 1,0,6 \rangle + t\langle 1,3,1 \rangle,\ t \in \mathbb{R}$

$x = 1+t,\ y = 3t,\ z = 6+t$

$x-1 = \frac{y}{3} = z-6$

(d) The line through the points $\overset{P_1}{(4,3,-1)}$ and $\overset{P_2}{(5,-1,0)}$



$\vec{v} = \overrightarrow{P_1P_2} = \langle 5-4, -1-3, 0-(-1) \rangle = \langle 1,-4,1 \rangle$ (or $\overrightarrow{P_2P_1}$ works)

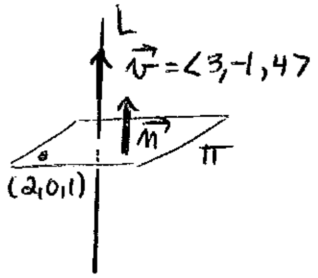
$\therefore$ L: $\vec{r} = \overbrace{\langle 4,3,-1 \rangle}^{(\text{or } \langle 5,-1,0 \rangle)} + t\langle 1,-4,1 \rangle,\ t \in \mathbb{R}$

$x = 4+t,\ y = 3-4t,\ z = -1+t$

$x-4 = \frac{y-3}{-4} = z+1$

10. Find an equation of the plane.

(a) The plane through the point $(2, 0, 1)$ and perpendicular to the line $x = 3t, y = 2 - t, z = 3 + 4t$



since $\vec{v} \parallel \vec{n}$, let $\vec{n} = \langle 3, -1, 4 \rangle$

$$3x - y + 4z + D = 0$$

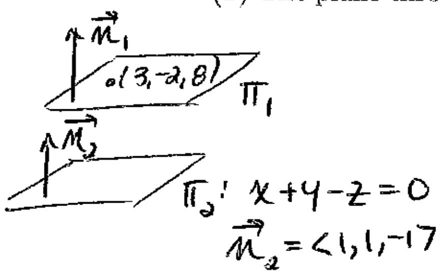
sub coordinates of point on plane in and solve for D:

$$3(2) - 0 + 4(1) + D = 0 \Rightarrow D = -10$$

$\therefore$ equation of plane is

$$3x - y + 4z - 10 = 0$$

(b) The plane through the point $(3, -2, 8)$ and parallel to the plane $z = x + y$



$\pi_1 \parallel \pi_2 \Rightarrow \vec{n}_1 \parallel \vec{n}_2$ so let $\vec{n}_1 = \langle 1, 1, -1 \rangle$.

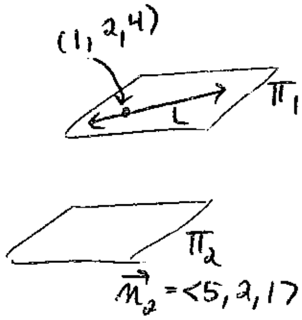
$$\pi_1 : x + y - z + D = 0$$

sub in point on π_1 to solve for D:

$$3 + (-2) - 8 + D = 0 \Rightarrow D = 7$$

$\therefore \pi_1 : x + y - z + 7 = 0$

(c) The plane that contains the line $x = 1 + t, y = 2 - t, z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$



$\Pi_1 \parallel \Pi_2 \Rightarrow \vec{n}_1 \parallel \vec{n}_2$ so let $\vec{n}_1 = \langle 5,2,1 \rangle$

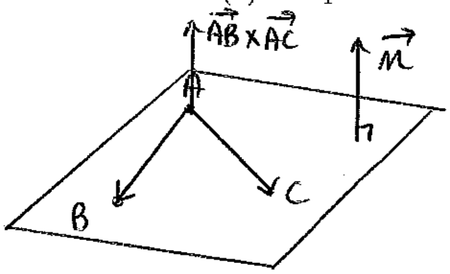
$\Pi_1 : 5x + 2y + z + D = 0$

sub point on line (+ on Π_1) in to solve for D:

$5(1) + 2(2) + 4 + D = 0 \Rightarrow D = -13$

$\therefore \Pi_1 : 5x + 2y + z - 13 = 0$

(d) The plane through the points $\overset{A}{(3,0,-1)}$, $\overset{B}{(-2,-2,3)}$, and $\overset{C}{(7,1,-4)}$



$\vec{AB} = \langle -2-3, -2-0, 3-(-1) \rangle = \langle -5, -2, 4 \rangle$

$\vec{AC} = \langle 7-3, 1-0, -4-(-1) \rangle = \langle 4, 1, -3 \rangle$

let $\vec{n} = \vec{AB} \times \vec{AC} = \langle 2, 1, 3 \rangle$

$\cancel{-5} \quad \cancel{4}$
$-2 \quad 1$
$4 \quad -3$
$-5 \quad 4$
$-2 \quad 1$
$\cancel{4} \quad \cancel{-3}$

$x = (-2)(-3) - (4)(1) = 2$
$y = (4)(4) - (-5)(-3) = 1$
$z = (-5)(1) - (-2)(4) = 3$

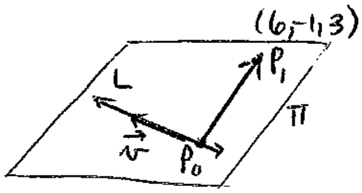
$\therefore \Pi : 2x + y + 3z + D = 0$

sub any point on plane to solve for D:

$2(3) + 0 + 3(-1) + D = 0 \Rightarrow D = -3$

$\therefore \Pi : 2x + y + 3z - 3 = 0$

(e) The plane that passes through the point $(6,-1,3)$ and contains the line with symmetric equations $\frac{x}{3} = y+4 = \frac{z}{2}$



L: $\vec{v} = \langle 3,1,2 \rangle$

point on line : $(0,-4,0)$

$\vec{P_0P_1} = \langle 6-0, -1-(-4), 3-0 \rangle = \langle 6,3,3 \rangle$

since $\vec{v}$ and $\vec{P_0P_1}$ lie in the plane,

$\vec{v} \times \vec{P_0P_1}$ will be perpendicular to the plane

$\vec{v} \times \vec{P_0P_1} = \langle -3,3,3 \rangle = -3\langle 1,-1,-1 \rangle$

$$\begin{array}{cc} \cancel{3} & \cancel{6} \\ 1 & 3 \\ 2 & 3 \\ 3 & 6 \\ 1 & 3 \\ \cancel{2} & \cancel{3} \end{array} \qquad \begin{array}{l} x = (1)(3)-(2)(3) = -3 \\ y = (2)(6)-(3)(3) = 3 \\ z = (3)(3)-(1)(6) = 3 \end{array}$$

Let $\vec{n} = \langle 1,-1,-1 \rangle$ (or $\langle -3,3,3 \rangle$)

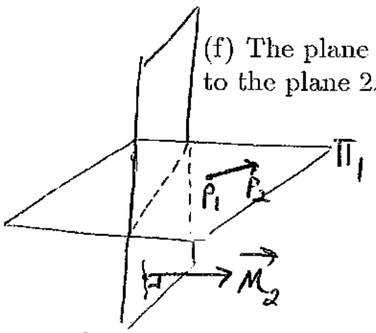
$x - y - z + D = 0$

$6-(-1)-(3)+D = 0 \Rightarrow D = -4$

$\therefore \boxed{\Pi : x-y-z-4=0}$

(f) The plane that passes through the points $(0,-2,5)$ and $(-1,3,1)$ and is perpendicular to the plane $2z = 5x+4y$

P_1 P_2



$\Pi_2 : 5x+4y-2z=0$

$\vec{n_2} = \langle 5,4,-2 \rangle$

$\vec{P_1P_2} = \langle -1-0, 3-(-2), 1-5 \rangle = \langle -1,5,-4 \rangle$

$\vec{n_2} \parallel \Pi_1$ and $\vec{P_1P_2} \parallel \Pi_1$ so

$\vec{n_2} \times \vec{P_1P_2} \perp \Pi_1$

$\vec{n_2} \times \vec{P_1P_2} = \langle -6, 22, 29 \rangle$

$$\begin{array}{cc} 4 & 5 \\ -2 & -4 \\ 5 & -1 \\ 4 & 5 \end{array} \qquad \begin{array}{l} x = (4)(-4)-(-2)(5) = -6 \\ y = (-2)(-1)-(5)(-4) = 22 \\ z = (5)(5)-(4)(-1) = 29 \end{array}$$

$\therefore \Pi_1 : -6x+22y+29z+D=0$

$-6(0)+22(-2)+29(5)+D=0 \Rightarrow D = -101$

$\therefore \Pi_1 : -6x+22y+29z-101=0$

11. Determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them. (Round to one decimal place.)

(a) $\pi_1 : x + 4y - 3z = 1$, $\pi_2 : -3x + 6y + 7z = 0$

$\vec{n_1} = \langle 1, 4, -3 \rangle \qquad \vec{n_2} = \langle -3, 6, 7 \rangle = -3 \langle 1, -2, -\frac{7}{3} \rangle$

$\vec{n_1} \neq k\vec{n_2}$ for some $k \Rightarrow$ not parallel

$\vec{n_1} \cdot \vec{n_2} = (1)(-3) + (4)(6) + (-3)(7) = -3 + 24 - 21 = 0 \Rightarrow \vec{n_1} \perp \vec{n_2}$

$\therefore \ \Pi_1 \perp \Pi_2$

(b) $\pi_1 : 9x - 3y + 6z = 2$, $\pi_2 : 2y = 6x + 4z$

$\vec{n_1} = \langle 9, -3, 6 \rangle = 3\langle 3, -1, 2 \rangle \quad \vec{n_2} = \langle 6, -2, 4 \rangle = 2\langle 3, -1, 2 \rangle$

$\vec{n_1} = \frac{3}{2}\vec{n_2} \Rightarrow \vec{n_1} \parallel \vec{n_2}$ and so $\Pi_1 \parallel \Pi_2$

(c) $\pi_1 : x - y + 3z = 1$, $\pi_2 : 3x + y - z = 2$

$\vec{n_1} = \langle 1, -1, 3 \rangle \qquad \vec{n_2} = \langle 3, 1, -1 \rangle \qquad |\vec{n_1}| = |\vec{n_2}| = \sqrt{11}$

$\vec{n_1} \neq k\vec{n_2}$ for some $k \Rightarrow$ planes are not parallel

$\vec{n_1} \cdot \vec{n_2} = (1)(3) + (-1)(1) + (3)(-1) = -1 \Rightarrow$ planes are not $\perp$

$$\theta = \cos^{-1}\left(\frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right) = \cos^{-1}\left(\frac{-1}{11}\right) \approx 95.2^\circ$$

↑ angle between planes

(d) $\pi_1 : 5x + 2y + 3z = 2$, $\pi_2 : y = 4x - 6z$

$\vec{n_1} = \langle 5, 2, 3 \rangle \qquad \vec{n_2} = \langle 4, -1, -6 \rangle$

$\vec{n_1}$ is not parallel to $\vec{n_2}$

$\vec{n_1} \cdot \vec{n_2} = (5)(4) + (2)(-1) + (3)(-6) = 0 \Rightarrow \vec{n_1} \perp \vec{n_2}$

$\therefore \ \Pi_1 \perp \Pi_2$

12. Determine whether each statement is true or false. Explain your reasoning.

(a) $\vec{a} \times \vec{a} = \vec{0}$ for any vector $\vec{a} = < a_1, a_2, a_3 >$

TRUE.

$|\vec{a} \times \vec{a}| = |\vec{a}||\vec{a}| \sin 0^\circ = 0 \implies \vec{a} \times \vec{a} = \vec{0}$

(b) The line through $(-2, 4, 0)$ and $(1, 1, 1)$ is perpendicular to the line through $(2, 3, 4)$ and $(3, -1, -8)$.

$\vec{v_1} = \langle 1-(-2), 1-4, 1-0 \rangle = \langle 3, -3, 1 \rangle$

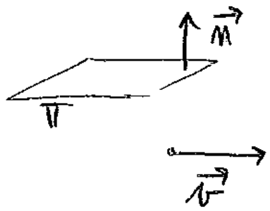
$\vec{v_2} = \langle 3-2, -1-3, -8-4 \rangle = \langle 1, -4, -12 \rangle$

$\vec{v_1} \cdot \vec{v_2} = (3)(1) + (-3)(-4) + (1)(-12) = 3 \neq 0$

$\therefore$ FALSE

(c) The vector $< 3, -1, 2 >$ is parallel to the plane $6x - 2y + 4z = 1$.

$\vec{v}$ $\qquad\qquad$ $\vec{n} = \langle 6, -2, 4 \rangle$



$\langle 3, -1, 2 \rangle \cdot \langle 6, -2, 4 \rangle = (3)(6) + (-1)(-2) + (2)(4) = 28 \neq 0$

$\therefore$ FALSE

If $\vec{v} \parallel \pi$, then $\vec{v} \cdot \vec{n} = 0$

13. Multiple choice. Clearly circle the one correct answer.

(a) Which of the following are **possible** to calculate?

$$\text{(I) } \vec{a} \times \underbrace{(\vec{b} \cdot \vec{c})}_{\text{scalar}} \ \times \qquad \text{(II) } \vec{a} \cdot \underbrace{(\vec{b} \times \vec{c})}_{\text{vector}} \ \checkmark \qquad \text{(III) } \vec{a} \times \underbrace{(\vec{b} \times \vec{c})}_{\text{vector}} \ \checkmark$$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

(b) Which of the following is/are true?

(I) If $\vec{a} \cdot \vec{b} = 0$, then $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$. ✗
(II) If $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$. ✗
(III) If $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$. ✓

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ ... If $\vec{a} \cdot \vec{b} = 0$, then either $|\vec{a}| = 0$, $|\vec{b}| = 0$, or $\theta = 90°$

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ ... If $\vec{a} \times \vec{b} = \vec{0}$ (and so $|\vec{a} \times \vec{b}| = 0$), then either $|\vec{a}| = 0$, $|\vec{b}| = 0$ or $\theta = 0°$ or $180°$

If $\vec{a} \cdot \vec{b} = 0$ <u>AND</u> $\vec{a} \times \vec{b} = \vec{0}$, then ① $|\vec{a}| = 0$ or ② $|\vec{b}| = 0$ or ③ $\cos\theta = 0$ AND $\sin\theta = 0$ $\Rightarrow \theta = 90°$ AND $\theta = 0°$ or $180°$

since option ③ is impossible, either ① or ② must be true, ie, $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

(A) none	(B) I only	(C) II only	(D) III only
(E) I and II	(F) I and III	(G) II and III	(H) all three

THE END