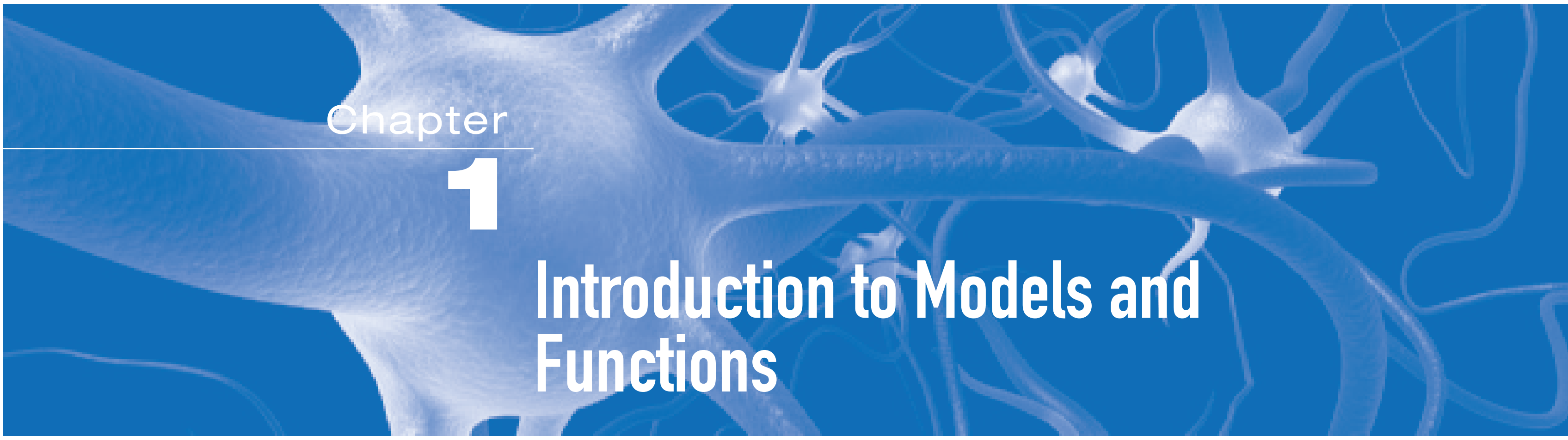

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Chapter 1

Introduction to Models and Functions

This chapter opens by answering an obvious and easy question—why do we need mathematics in the life sciences? We list some (of many!) questions from biology, health sciences, and elsewhere that are answered—using mathematics—in this book. As well, we mention a few present-day research problems that have very little chance of being solved or fully understood without mathematics.

We introduce the main tools needed to study biology using mathematics: **models** and **functions.** A **model** is a collection of mathematical objects (such as functions and equations) that allows us to interpret biological problems in the language of mathematics. Biological phenomena are often described by measurements: a set of numeric values with units (such as kilograms or metres). Many relations between measurements are described by **functions,** which assign to each input value a unique output value.

After talking about what constitutes a mathematical model and presenting a few examples, we briefly **review functions** and their properties. In this chapter, we discuss the **domain and range** and the **graph of a function, algebraic operations with functions, composition of functions,** and **inverse functions.** We build new functions from old using shifting, scaling, and reflections. We catalogue important elementary functions and note their properties. The reader who is familiar with functions might skip this material and move to the next chapter.

We introduce the four approaches—algebraic, numeric, geometric, and verbal—that we use throughout the book to discuss functions, their properties, and their applications.

In the last section we discuss aspects of logical reasoning that we need to follow and to understand mathematical expositions. As well, we contrast the language used in mathematics and in the life sciences, in order to motivate learning an important skill—communicating scientific facts, ideas, and results across disciplines.

1.1 Why Mathematics Matters

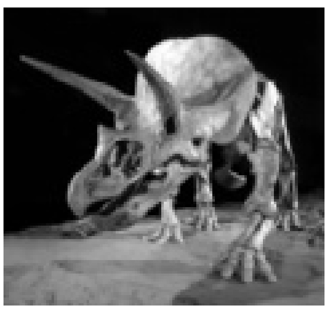

FIGURE 1.1.1

Complete skeleton of a triceratops, Royal Tyrrell Museum
Photo courtesy of the Royal Tyrrell Museum, Drumheller, Alberta

In the summer of 2012, a team of palaeontologists from the Royal Tyrrell Museum ("Canada's dinosaur museum") in Drumheller, Alberta, unearthed the skeleton of a large triceratops, a herbivorous dinosaur that lived in what is now North America some time between 68 and 65 million years ago. An adult triceratops measured 8–9 m in length and about 3 m in height, and weighed between 6,000 and 12,000 kg (Figure 1.1.1).

No human has ever seen a living dinosaur, so how do we know all this?

Estimates of age, size, weight, and many other quantities are obtained *using mathematics*, based on the data collected from the bones and from the site where they were found. Among other techniques, potassium-argon dating (see the note following Example 2.2.13) is used to compute the time interval when triceratops lived on Earth. By counting the growth lines in the MRI scan of certain bones (unlike an X-ray, an MRI image is *calculated*), researchers can determine how old the dinosaur was when it died. Then, using the formula (adapted from G. M. Erickson, K. C. Rogers, and S. A. Yerby, Dinosaurian growth patterns and rapid avian growth rates. *Nature*, 412 (429–433), 2001)

$$M = \frac{12,000}{1 + 2.9e^{-0.87(t-7.24)}} \qquad \textbf{(1.1.1)}$$

we can find an approximation of the body mass, M (in kilograms), of the triceratops based on the age at death, t (in years).

Allometry—a branch of life sciences—is the study of numeric relationships between quantities associated with human or animal organisms. For instance, the allometric formula (adapted from M. Benton, and D. Harper, *Basic Palaeontology*, Harlow, U.K.: Addison Wesley Longman, 1997)

$$\mathrm{Sk} = 0.49\mathrm{Sp}^{0.84} \qquad \textbf{(1.1.2)}$$

relates the skull length, Sk, of a larger dinosaur to its spine length, Sp (both measured in metres). From the triceratops' vertebrae found at the Drumheller site, researchers could figure out its spine length and then use formula (1.1.2) to calculate the size of its skull. (Further examples of allometric relationships can be found in Examples 2.1.12 to 2.1.16 and Example 5.5.10.)

This example, and many more that we will encounter in this book, echo this important message:

> Mathematics is an indispensable tool for studying life sciences. It deepens our understanding of life science phenomena and helps us to figure out the answers to questions that would otherwise be hard (or impossible) to find.

To further emphasize this message, we give a sample of questions that we will discuss and answer—using mathematics—in this book. Needless to say, we view mathematics in its broadest sense, i.e., including probability, statistics, numeric techniques and simulations, and computer programming.

- My body mass index is 27 (above normal range). How much weight should I lose to lower it to 24 (healthy weight)? My body mass index is 16 (below normal range). How much weight should I gain to bring it to 18 (healthy weight)? (See Example 2.1.11.)
- According to the Statistics Canada 2011 Census, about 33.5 million people lived in Canada in May 2011 (exactly 33,476,688 people were enumerated in the census). How many people will live in Canada in 2021? (See Examples 2.1.10 and 2.2.14.)
- If a student consumes one alcoholic drink (12 oz of beer, or 5 oz of white wine, or 1.5 oz of tequila or vodka) every hour, how much alcohol will be in that student's body after five hours? How long will it take the student to sober up? (See Section 3.3, in particular Examples 3.3.6–3.3.8.)
- How do forensic pathologists identify the location of impact (say, from a bullet) by analyzing blood splatters on the floor? (Read Example 2.3.15.)
- Which part of a skeleton grows faster: the skull or the spine? (See Example 5.5.10.)
- What is the surface area of Lake Ontario? (This information is needed, for instance, when scientists study the impact of pollutants on lake fauna; see Example 7.6.6.)
- Scientists believe that the fossils found in the Burgess Shale Formation in British Columbia are about 505 million years old. The Joggins Fossil Cliffs in Nova Scotia contain fossils from the so-called coal age of Earth's history, about 310 million years ago. How were these estimates obtained? A tree trunk (Figure 1.1.2) was unearthed near the city of Kaitaia in New Zealand. How long ago did the tree die? (See the note following Example 2.2.13.)

FIGURE 1.1.2

Ancient kauri tree trunk (unearthed and placed upright for display)
Miroslav Lovric

- An MRI (magnetic resonance imaging) scan shows that a smaller blood vessel branches off the right coronary artery at an angle of 50 degrees. Is there a reason for medical concern? (Read Example 6.1.18.)
- Long bones in mammals (such as the femur) are hollow, filled with blood cell–producing marrow. Although lightweight, they are strong enough to support the entire body, enabling it to move in various ways. However, under continuous stress (often identified in athletes) or due to an acute event (such as a fall or a car crash), a femur can break. Can we somehow grow a stronger femur by making its walls thicker? (See Section 6.2.)
- How much valuable time is saved if a breast cancer is detected in a mammogram compared to a clinical breast examination detection? (See Example 2.2.15.)

Formulas (1.1.1) and (1.1.2), which describe life science quantities and phenomena using mathematical objects (in this case formulas involving exponential and power functions) are said to constitute a **mathematical model** (or just a **model,** as is common in practice). For instance, formula (1.1.1) **models** the relationship between age at death and body mass for larger dinosaurs. Formula (1.1.2) is a **model** for a relationship between skull length and spine length for large dinosaurs. We will say more about models in the next section.

In this book we study numerous ways of describing how populations (of cells, bacteria, animals, or humans) change. The simplest model, which uses elementary mathematics, assumes that the birth and the death rates are constant. If the birth rate is larger than the death rate, the model implies that the population will grow exponentially. Of course, beyond a certain point, this is neither realistic nor possible for any population.

To make this model better mimic reality, we introduce various modifications: for instance, we can make the birth and the death rates change over time, we can include the carrying capacity (carrying capacity is the largest number of individuals that can live in an ecosystem), or we might need to include a term that accounts for the minimum number of individuals needed for the population to avoid extinction. Having made some or all of these modifications to our model, we realize that *we need to know more mathematics* in order to work with it.

This is not all—we might need to add terms that account for harvesting and seasonal changes in the population size. As well, it might be necessary to include the effects of a disease, a natural disaster, or another random event that might affect the population. For all this, we need to know *even more mathematics.* Hence another important message:

> As the model—the description of a life sciences phenomenon using mathematics—moves closer to reality, it also becomes more complex, and more mathematics is needed to work with it.

In other words, as we learn more math, we are able to probe deeper into a problem, understand it better, gain new insights, and obtain more meaningful results and answers. To further stimulate interest in studying life sciences and mathematics together, we list several problems that are the topics of present research:

- What are the risks to the indigenous fish populations in the Great Lakes from the new species of fish brought in the tanks of large cargo ships?
- What are the distinct features of the trafficking of eosinophils as they migrate from bone marrow to the blood and, ultimately, to the lungs? How can this enhance our understanding of certain aspects of the development of allergic asthma? (Eosinophils are white blood cells, important components of our immune system, defending it against parasites and certain infections. Allergic

asthma is a disease of the airways that develops as a consequence of an immune-inflammatory response to allergen exposure (such as dust, pollen, or various drugs), causing inflammation in the lungs.)

- According to the Director of Biodiversity Programs at the Royal Botanical Gardens (RBG) in Hamilton, Ontario, there is a need to cut "an apparent overpopulation of deer wintering on its lands." An aerial survey of the part of the RBG lands in 2010 identified 267 deer, which is deemed (by the Ontario Ministry of Natural Resources) to be six to nine times the desired number of deer in the area. Is the ministry right? What is the carrying capacity of the RBG lands, i.e., how many deer should be removed (culled or relocated) from there?
- Due to long exposures to zero or near-zero gravity, astronauts working in the International Space Station suffer from spaceflight osteopenia (bone loss; on average, they lose about 1% of their bone mass per month spent in space). On the other hand, sea urchins are known to continue the production of calcium, unaffected by the lack of gravity. By modelling the growth and development of calcium plates in the skeletons of the urchins, researchers are trying to shed more light on the dynamics of calcium recycling, hoping to reduce the effects of osteopenia in humans.
- In order to better understand the pathophysiology of hydrocephalus (potentially brain-damaging buildup of fluid in the skull), researchers are studying the interaction between the cerebrospinal fluid and the brain tissue. Present efforts are focused on using partial differential equations (we will study differential equations in this book) to gain new insights into this interaction.

Dear Student:

This book gives you an opportunity to learn mathematics and to see how it is used in the wide spectrum of applications in the life sciences. To see applications in action (and to get more from them!) you need to understand the underlying math concepts, formulas, and algorithms. This is why you will find a large number of fully solved examples, as well as exercises, ranging from easy and routine to more complex, theoretical, and challenging. Work on as many of them as you can.

Learning mathematics is not easy. Like everything you really care for, it requires seriousness, dedication, a significant amount of time, and lots of hard work. But in the end, it will be worth it!

> No subject teaches logical thinking, develops analytic and problem-solving skills, and demonstrates how to deal with complex problems better or more effectively than math.

Have you ever wondered why math majors score at or near the top in all standardized tests, including MCAT, GMAT, and LSAT?

1.2 Models in Life Sciences

Living systems, from cells to organisms to ecosystems, are characterized by change and dynamics. Living things grow, maintain themselves, and reproduce. Even remaining the same requires dynamical responses to a changing environment. Understanding the mechanisms behind these dynamics and deducing their consequences is crucial to understanding biology, biochemistry, ecology, epidemiology, physiology, population genetics, and many other life sciences.

This dynamical approach is necessarily mathematical because describing dynamics requires quantifying measurements. What is changing? How quickly is it changing? What is it changing into?