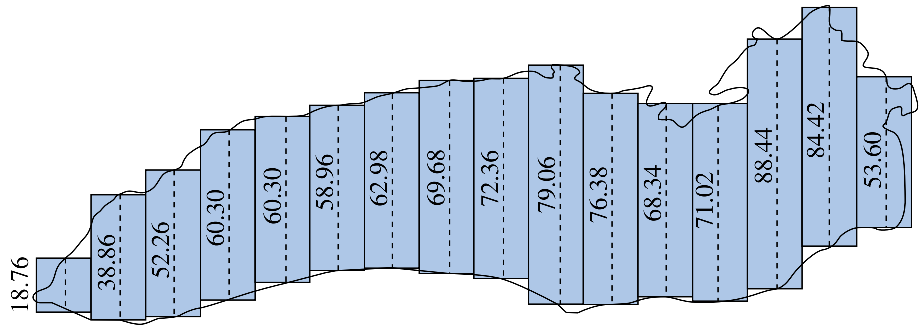

FIGURE 7.6.56c
Approximating rectangles

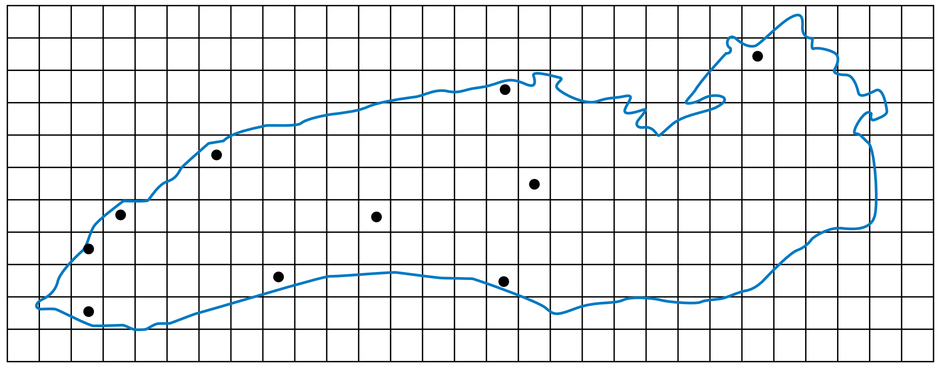

FIGURE 7.6.57
Counting dots to estimate the area of Lake Ontario

In our case, we counted 137 squares. The size of the grid (in this case) is 12 km, so the estimate for the area is

$$137 \cdot 12^2 = 137 \cdot 144 = 19{,}728 \text{ km}^2$$

Note that every square that is counted contributes 144 km^2 to the total area. So our decision on which squares to count (or not) around the edges could significantly affect the estimate.

Integrals and Volumes

We *approximated* the area of a (two-dimensional) region between the graph of a function and the x-axis by using simple geometric objects (rectangles). Then, by computing the limit of approximations by rectangles we arrived at the *exact* value for the area (Section 7.3).

We compute the volume of a three-dimensional solid in space in essentially the same way. This time, the simple three-dimensional objects we use are cylinders. A **right cylinder** is a solid bounded by a two-dimensional region (called the **base**) and an identical region (called the **top**) in a plane parallel to the base. The cylinder contains all points on the line segments perpendicular to the base (hence "right") that connect the base and the top. The distance between the base and the top is the **height** of the cylinder (labelled h in Figure 7.6.58). In Figure 7.6.58a we show a right circular cylinder (since its base is a disk). Two more cylinders are drawn in Figures 7.6.58b and c. The volume of a cylinder is computed by multiplying the base area by the height. Thus, the volume of the right circular cylinder of radius r and height h is $\pi r^2 h$.

Let's see how cylinders are used to approximate three-dimensional regions, thus producing an approximation for their volume.

Example 7.6.7 Volume of a Heart Chamber

In Figure 7.6.59a we see a sonogram (an ultrasound image) of a human heart, showing its structures—in particular, the four chambers. The information about the volume of

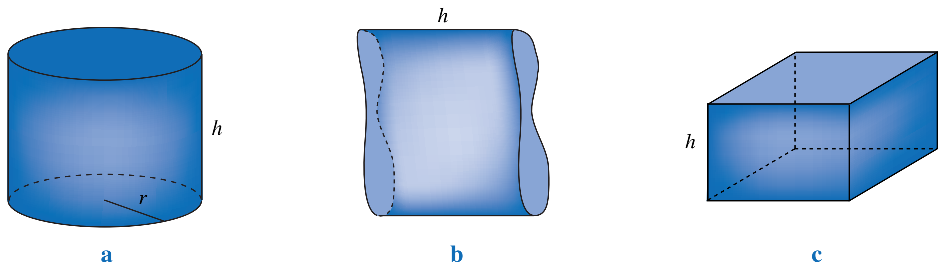

FIGURE 7.6.58
Cylinders

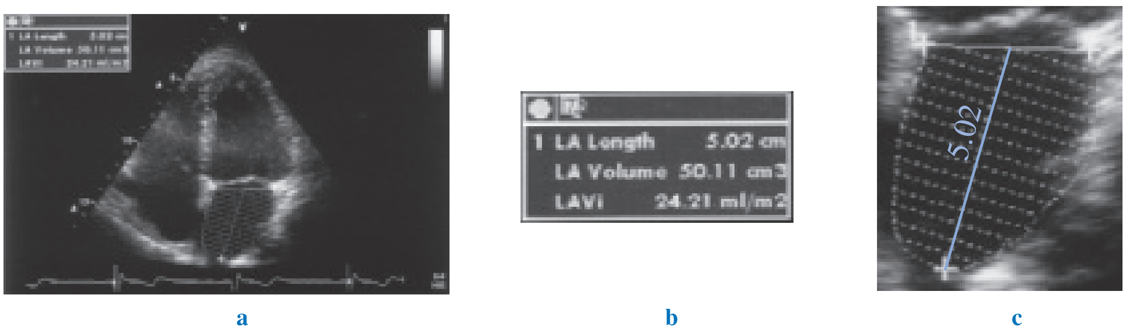

FIGURE 7.6.59
Ultrasound image of a human heart
Miroslav Lovric

the chambers is essential in determining certain features of the blood flow in and out of the heart. The zoom-in of the upper left corner of the sonogram reads "LA Volume 50.11 cm^3" (Figure 7.6.59b; LA stands for left atrium chamber, shown on the bottom right of the sonogram image). How was the volume calculated?

With a computer mouse, an ultrasound technician outlines the shape of the chamber. (Note that it is not possible to identify where exactly the boundary of the chamber is located.) When the mouse button is released, the computer draws the grid shown in Figures 7.6.59a and c. The measurement "LA Length 5.02 cm" (Figure 7.6.59b) is the distance across the chamber shown in Figure 7.6.59c.

The grid consists of 15 strips of thickness $5.02/15 \approx 0.335$ cm. By using the lengths of the parallel dashed lines as diameters, we draw right circular cylinders (Figures 7.6.60a, b), obtaining an approximation of the chamber by circular cylinders (Figure 7.6.60c). The height of each cylinder is 0.335 cm.

The diameter of the cylinder at the bottom of the chamber (Figure 7.6.60a) is measured to be 2 cm; thus, its volume is $\pi(1)^2(0.335) \approx 1.052$ cm^3. The diameter of the cylinder above it is 2.5 cm, and its volume is $\pi(1.25)^2(0.335) \approx 1.644$ cm^3. The remaining diameters (from bottom to top) are 3.0, 3.21, 3.43, 3.57, 3.93, 4.07, 4.29, 4.29, 4.29, 4.14, 4.07, 3.43, and 2.14 cm, respectively. Computing the corresponding volumes and then adding up the volumes of all 15 cylinders, we obtain 50.342 cm^3 as an approximation of the volume of the heart chamber.

Our estimate is very close to the approximation of 50.11 cm^3 produced by the computer and shown in the sonogram. The discrepancy is due to the errors we made in measuring the diameters. ▲

We use the same idea to find the volume of any three-dimensional solid S (Figure 7.6.61). First we divide S into n pieces and approximate each piece with a cylinder. By adding up the volumes of all cylinders (thus forming a Riemann sum!), we obtain an approximation for the volume of S. Increasing the number of cylinders (analogous to the case of the area in Section 7.3) generates better and better approximations. Finally, in the limit as $n \to \infty$, we obtain the exact value of the volume.

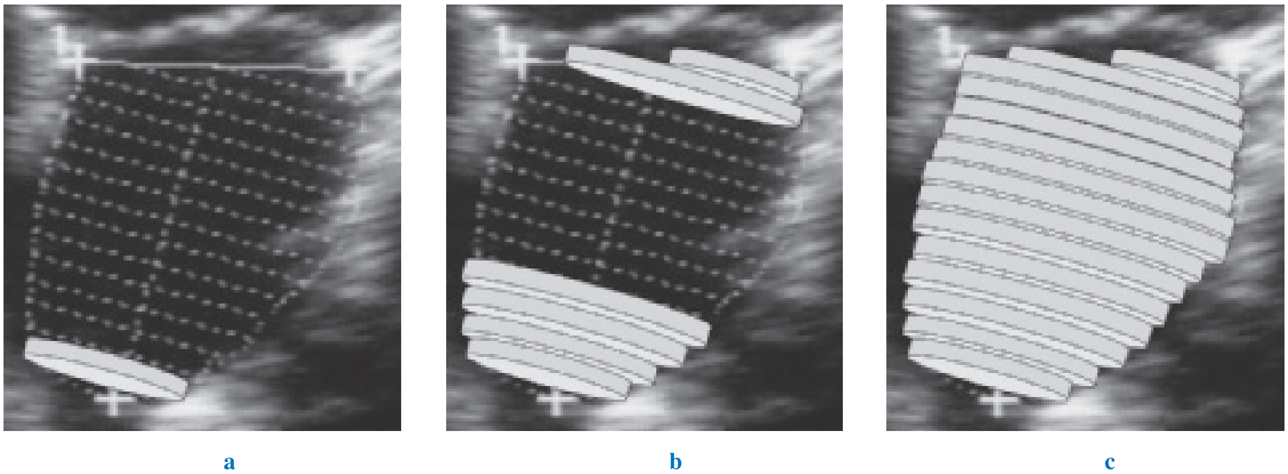

FIGURE 7.6.60

Approximating cylinders
Miroslav Lovric

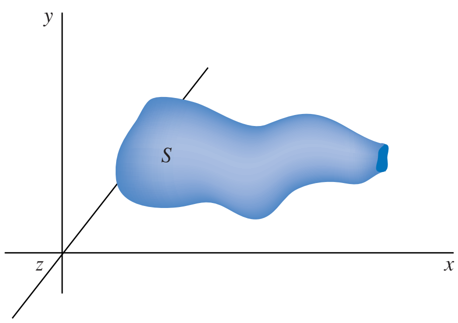

FIGURE 7.6.61

Three-dimensional solid S

Now the details. Assume that S lies between two parallel planes that are perpendicular to the x-axis and intersect it at $x = a$ and $x = b$ (Figure 7.6.62a). Pick any location x between a and b, and cut S with a plane perpendicular to the x-axis passing through x. This way, we obtain a two-dimensional region R_x called the *cross-section* of the solid S (Figure 7.6.62a). Denote by $A(x)$ the area of R_x (we learned how to calculate areas earlier in this chapter). As cross-sections vary with x, so does their area—thus A is indeed a function of x; it is defined on $[a, b]$.

Divide the interval $[a, b]$ into n subintervals (n is a positive integer)

$$[a = x_1, x_2], [x_2, x_3], \ldots, [x_i, x_{i+1}], \ldots, [x_n, x_{n+1} = b]$$

of length $\Delta x = (b - a)/n$ and form the cross-sections of S at points $x_1, x_2, \ldots, x_n$. "Thicken" each cross-section to make a cylinder whose base is the cross-section and whose height is Δx; see Figure 7.6.62b, where we thickened the cross-section at x_i. The volume of the cylinder thus obtained is the base area, $A(x_i)$, multiplied by the height, Δx. By adding up the volumes of all thickened cylinders,

$$V_n = A(x_1)\Delta x + A(x_2)\Delta x + A(x_3)\Delta x + \cdots + A(x_n)\Delta x = \sum_{i=1}^{n} A(x_i)\Delta x$$

we obtain an approximation of the volume, V, of the solid S. (Note that V_n is the left Riemann sum for the function $A(x)$ on the given interval.)

It can be proven that, as n increases (i.e., as the approximating cylinders become thinner and thinner), the approximations V_n become better and better. We now define the volume as the limit of V_n. Keep in mind that, at the same time, we are computing the limit of Riemann sums—thus obtaining a definite integral.

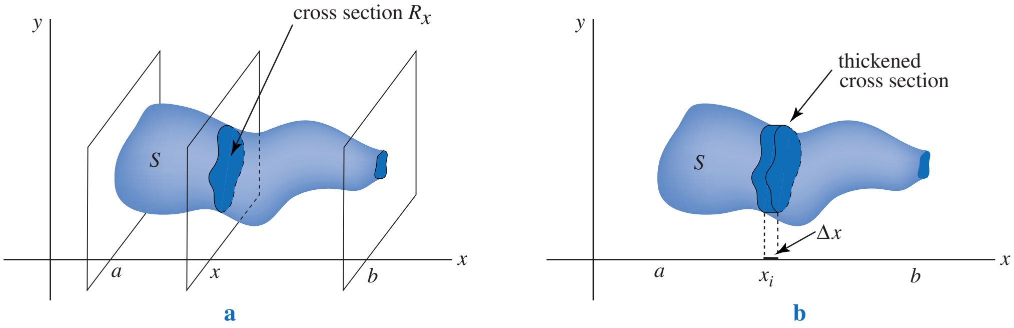

FIGURE 7.6.62

Approximating the solid S with cylinders

Definition 7.6.1 Assume that S is a solid three-dimensional region that lies between $x = a$ and $x = b$. Denote by $A(x)$ the area of the cross-section of S by the plane perpendicular to the x-axis that passes through x. Assume that $A(x)$ is continuous on $[a, b]$. Then the **volume** V of S is given by

$$V = \lim_{n\to\infty} V_n = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x = \int_a^b A(x)dx$$

provided that the limit exists (i.e., is equal to a real number).

Note that Definition 7.6.1 is essentially—with necessary technical adjustments—the definition of the area (Definition 7.3.1) moved one dimension higher.

In conclusion, to compute the volume of a solid S that lies between $x = a$ and $x = b$ we integrate the areas of its cross-sections from a to b. We now illustrate this idea in several examples.

Example 7.6.8 Volume of a Cylinder

Assume that the cross-sectional area of the cylinder in Figure 7.6.58b is A. In Figure 7.6.63a we redrew the cylinder, placing it so that we can calculate its volume by integration. Since all cross-sections are identical, we find

$$V = \int_0^h A(x)dx = \int_0^h A\,dx = Ax\Big|_0^h = Ah$$

as claimed at the start of this section.

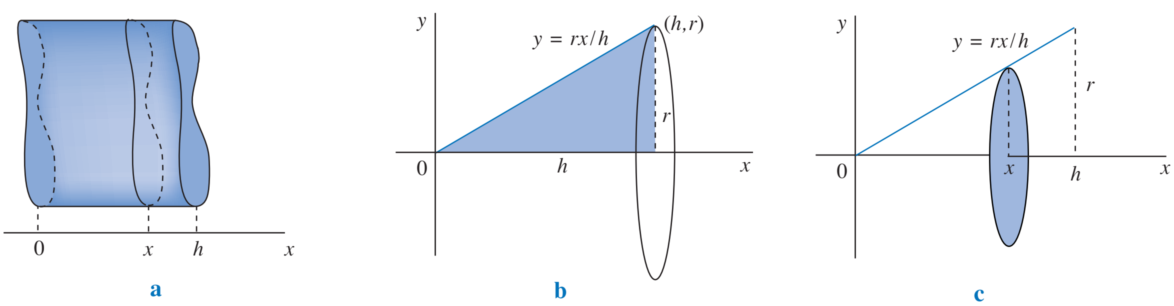

FIGURE 7.6.63

A cylinder and a cone

Example 7.6.9 Volume of a Cone

Find the volume of a cone of radius r and height h.

Look at Figure 7.6.63b. The line joining the point (h, r) to the origin has slope r/h, and so its equation is $y = rx/h$. The cone is now obtained by rotating the shaded triangular region about the x-axis. At a location x, the cross section is a disk of radius rx/h (Figure 7.6.63c). By Definition 7.6.1, the volume of the cone is

$$V = \int_0^h A(x)dx = \int_0^h \pi \left(\frac{rx}{h}\right)^2 dx = \pi \frac{r^2}{h^2} \int_0^h x^2 dx = \pi \frac{r^2}{h^2} \left(\frac{x^3}{3}\Bigg|_0^h\right) = \pi \frac{r^2}{h^2} h^3 = \pi r^2 h$$

The cone is an example of a **solid of revolution.** In general, we obtain a solid of revolution by rotating a two-dimensional region about an axis in space.

As a special case, the region R between the graph of a continuous function $f(x)$ and the x-axis, and between the vertical lines $x = a$ and $x = b$, is rotated about the x-axis (Figure 7.6.64a). The resulting solid of revolution, S, is shown in Figure 7.6.64b.

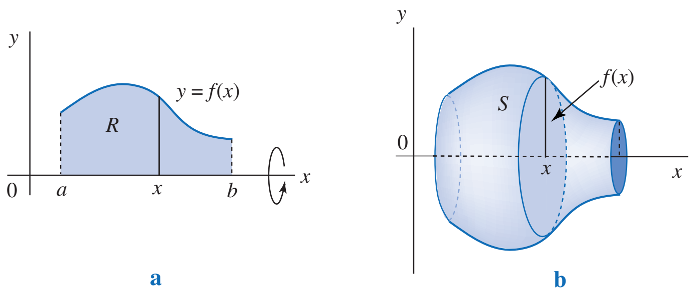

FIGURE 7.6.64
Solid of revolution

At a location x, the cross-section of S is a disk of radius $|f(x)|$ (in the case drawn in Figure 7.6.64a, $f(x) \geq 0$; in general, $f(x)$ could be negative). The area of the cross-section is $A(x) = \pi|f(x)|^2 = \pi[f(x)]^2$, and the volume of S is

$$V = \int_a^b A(x)dx = \pi \int_a^b [f(x)]^2\, dx$$

Example 7.6.10 Volume of a Sphere

Find the volume of a sphere of radius r.

From $x^2 + y^2 = r^2$ we obtain the equation of the semicircle $y = \sqrt{r^2 - x^2}$ in the upper half-plane. Denote by R the region enclosed by this semicircle and the x-axis. The rotation of R about the x-axis generates the sphere (Figure 7.6.65; so a sphere can be viewed as a solid of revolution). Its volume is

$$\begin{aligned} V &= \pi \int_{-r}^{r} [f(x)]^2\, dx = \pi \int_{-r}^{r} \left(r^2 - x^2\right)\, dx \\ &= \pi \left(r^2 x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r} \\ &= \pi \left(r^3 - \frac{r^3}{3}\right) - \pi\left(-r^3 + \frac{r^3}{3}\right) = \frac{4}{3}\pi r^3 \end{aligned}$$

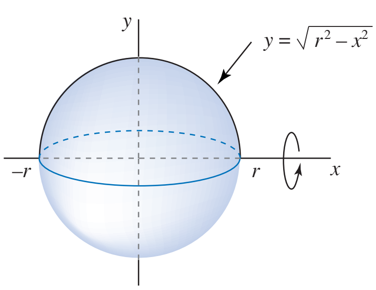

FIGURE 7.6.65
Sphere as a solid of revolution

Example 7.6.11 Volume of a Solid of Revolution Obtained by Rotation about the y-axis

Denote by R the bounded region in the first quadrant between the curves $y = e^x$ and $y = e$. Find the volume of the solid, S, obtained by rotating R about the y-axis.

We sketch the region R in Figure 7.6.66a. Because R is rotated about the y-axis, we consider the cross-sections perpendicular to the y-axis—and therefore to obtain the volume we need to integrate along the y-axis. For that reason, we view the curve $y = e^x$ as $x = \ln y$.

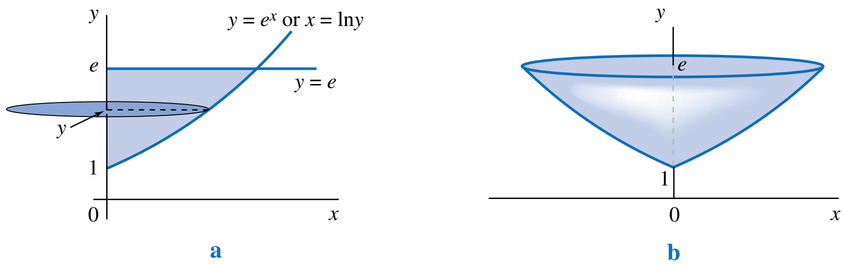

FIGURE 7.6.66

Integrating along the y-axis to calculate the volume

The cross-section at a location y is a disk of radius $\ln y$. Its area is $A(y) = \pi(\ln y)^2$, and the volume of S is

$$V = \pi \int_1^e A(y)\, dy = \pi \int_1^e (\ln y)^2\, dy$$

Using integration by parts (see Exercise 41 in Section 7.5), we obtain

$$\int (\ln y)^2\, dy = y(\ln y)^2 - 2y \ln y + 2y$$

The volume of S (shown in Figure 7.6.66b) is computed to be

$$V = \pi \left[y(\ln y)^2 - 2y \ln y + 2y \right] \Big|_1^e = \pi\,[(e - 2e + 2e) - 2] = \pi(e - 2)$$

Example 7.6.12 Volume of a Solid of Revolution

Find the volume of a solid S generated by rotating the region R from Example 7.6.11 about the x-axis.

This time, the cross-section at a location x is not a disk, but a washer (a disk from which a smaller concentric disk has been removed); see Figure 7.6.67a and b. The radius of the larger (outer) disk is constant and equal to e, and the radius of the smaller (inner) disk is e^x. The area of the washer is obtained by subtracting the area of the inner disk from the area of the outer disk:

$$A(x) = \pi(e)^2 - \pi\,(e^x)^2 = \pi\left(e^2 - e^{2x}\right)$$

The volume of S is

$$V = \pi \int_0^1 A(x) dx = \pi \int_0^1 \left(e^2 - e^{2x}\right) dx = \pi \left[e^2 x - \frac{1}{2} e^{2x} \right] \Big|_0^1 = \pi \left[\frac{1}{2} e^2 + \frac{1}{2} \right] = \frac{\pi}{2}\left(e^2 + 1\right)$$

The solid S is drawn in Figure 7.6.67c.

Integrals and Lengths

Area is a measurement of the size of two-dimensional regions, whereas volume determines the size of three-dimensional solids. **Length** measures the size of one-dimensional objects—line segments and curves.

The length of a straight line segment joining two points (x_1, y_1) and (x_2, y_2) in a plane is given by the **distance formula**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$