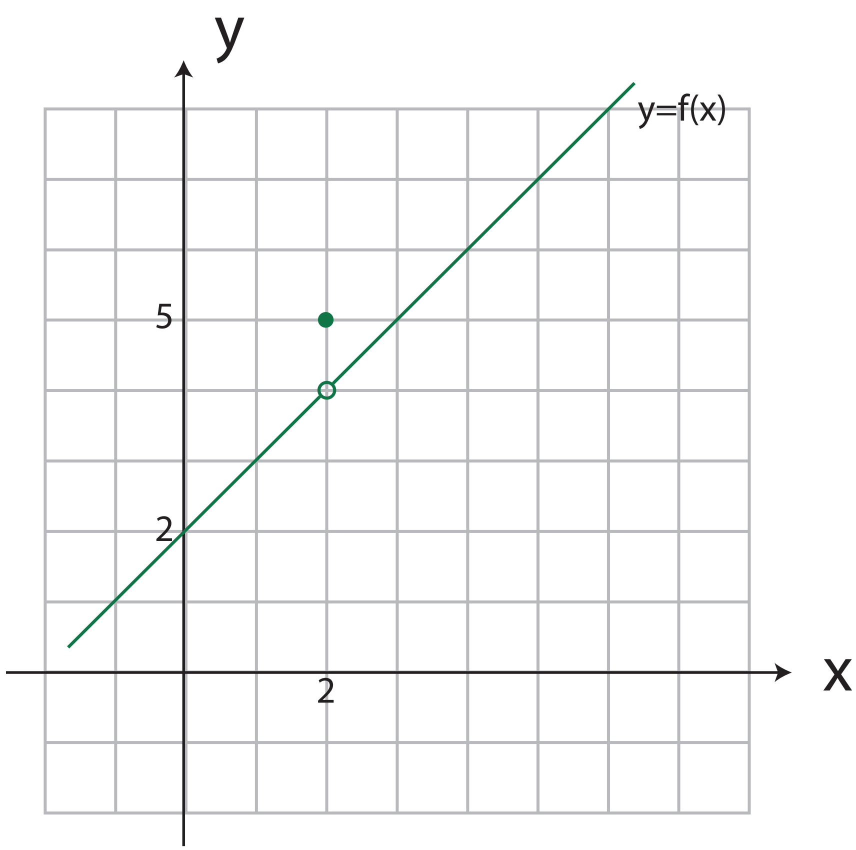
The Limit of a Function



$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$$

Notations:

$$f(2) = 5$$

means that the y-value of the function AT x=2 EQUALS 5

$$\lim_{x \to 2} f(x) = 4$$

means that the y-values of the function APPROACH 4 as x APPROACHES 2

The Limit of a Function

Definition:

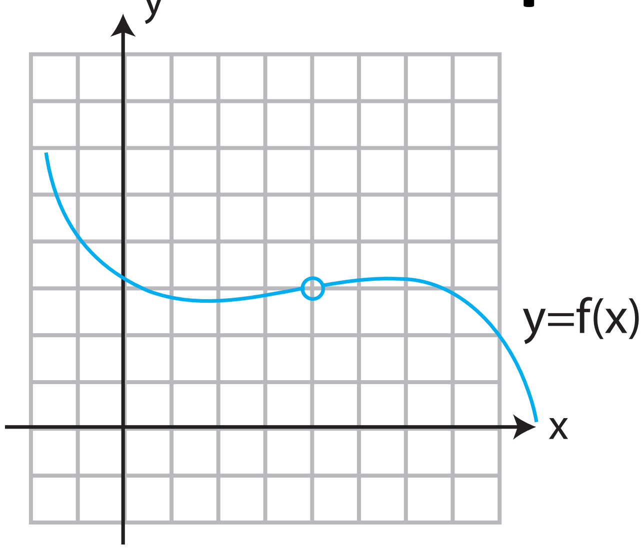
$$\lim_{x \to a} f(x) = L$$

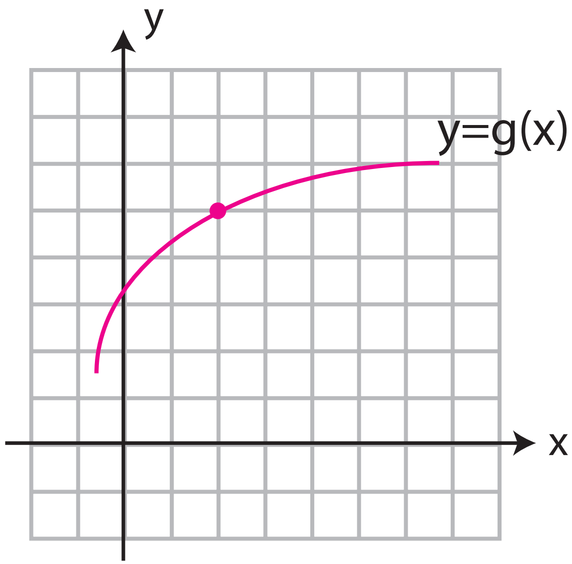
"the limit of f(x), as x approaches a, equals L"

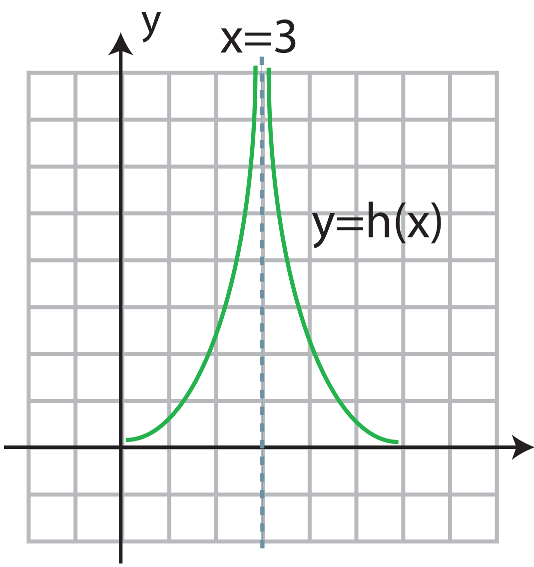
means that the values of f(x) can be made as close as we'd like to L by taking x sufficiently close to a, but not equal to a.

Limit of a Function

Some examples:







<u>Note</u>: f may or may not be defined at x=a. Limits are only asking how f is defined **NEAR** a.

Left-Hand and Right-Hand Limits

$$\lim_{x \to a^-} f(x) = L$$

means $f(x) \to L$ as $x \to a$ from the left $(x < a)$.

$$\lim_{x \to a^+} f(x) = L$$

means $f(x) \to L$ as $x \to a$ from the right $(x > a)$.

** The full limit exists if and only if the left and right limits both exist (equal a real number) and are the same value.

Left-Hand and Right-Hand Limits

Example:

Consider the function $f(x) = \begin{cases} 1 - x, & x < -1 \\ x^2, & -1 \le x \le 1 \\ x^{-1}, & x > 1 \end{cases}$

Determine the following limits, if they exist.

(a) $\lim_{x \to -1} f(x)$ (b) $\lim_{x \to 1} f(x)$

Evaluating Limits

We can evaluate the limit of a function in 3 ways:

1. Graphically
2. Numerically
3. Algebraically

Evaluating a Limit Numerically

Example:
Use a table of values to estimate the value of

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$$

x	f(x)
3.5	
3.9	
3.99	
4	undefined
4.01	
4.1	
4.5	

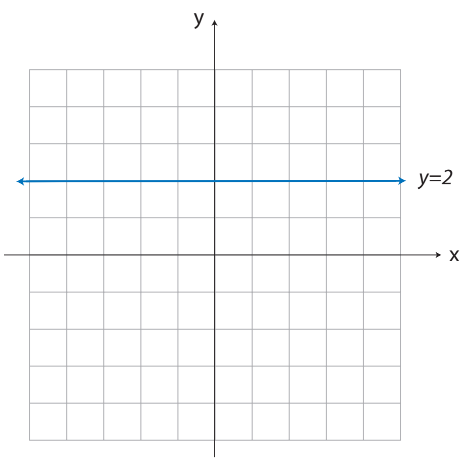
BASIC LIMITS

<u>Limit of a Constant Function</u>

$$\lim_{x \to a} c = c, \quad \text{where } c \in R$$

Example:

$$\lim_{x \to 3} 2 = 2$$

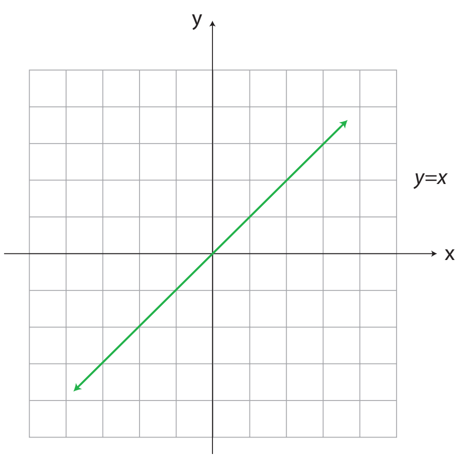


<u>Limit of the Identity Function</u>

$$\lim_{x \to a} x = a$$

Example:

$$\lim_{x \to 3} x = 3$$



LIMIT LAWS

Suppose that *c* is a constant and the limits

$$\lim_{x \to a} f(x) \text{ and } \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} \left[c f(x)\right] = c \lim_{x \to a} f(x)$

LIMIT LAWS

Continued…

4. $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} [f(x) \div g(x)] = \lim_{x \to a} f(x) \div \lim_{x \to a} g(x), \text{ if } \lim_{x \to a} g(x) \neq 0$

Evaluating Limits Algebraically

Example:

Evaluate the limit and justify each step by indicating the appropriate Limit Laws.

$$\lim_{x\to 1}(x^2 - 5x + 6)$$

$$= \lim_{x\to 1} x^2 - \lim_{x\to 1} 5x + \lim_{x\to 1} 6$$

$$= \lim_{x\to 1} x \cdot \lim_{x\to 1} x - 5\lim_{x\to 1} x + \lim_{x\to 1} 6$$

$$= (1)(1) - 5(1) + 6$$

$$= 2$$

Direct Substitution Property

From the previous slide, we have

$$\lim_{x \to 1} \underbrace{(x^2 - 5x + 6)}_{f(x)} = \underbrace{2}_{f(1)}$$

Notice that we could have simply found the value of the limit by plugging in x=1 into the function.

Direct Substitution Property

Direct Substitution Property:

If $f(x)$ is an **algebraic**, **exponential**, **logarithmic**, **trigonometric**, or **inverse trigonometric** function, and a is in the domain of $f(x)$, then

$$\lim_{x \to a} f(x) = f(a)$$

Evaluating Limits Algebraically

Evaluate each limit or state that it does not exist.

(a) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

(b) $\lim_{x \to 2} \frac{\frac{1}{2} - \frac{1}{x}}{x - 2}$

(c) $\lim_{x \to -1} \frac{x + 2}{x + 1}$

(d) $\lim_{x \to 1} \frac{1 - |x|}{x - 1}$