Infinite Limits

Example:
Use a table of values to estimate the value of

$$\lim_{x \to 0} \frac{1}{x}$$

x	f(x)
0.1	
0.01	
0.001	
0	undefined
-0.001	
-0.01	
-0.1	

Infinite Limits

Definition:

$$\lim_{x \to a} f(x) = \infty$$

“the limit of f(x), as x approaches a, is infinity”

means that the values of f(x) (y-values) increase **without bound** as x becomes closer and closer to a (from either side of a), but $x \neq a$.

Definition:

$$\lim_{x \to a} f(x) = -\infty$$

“the limit of f(x), as x approaches a, is negative infinity”

means that the values of f(x) (y-values) decrease **without bound** as x becomes closer and closer to a (from either side of a), but $x \neq a$.

Infinite Limits

Example:

Determine the infinite limit.

(a) $\lim_{x \to -1} \frac{x+2}{x+1}$

Note:
Since the values of these functions do not approach a real number L, these limits **do not exist**.

(b) $\lim_{x \to \pi^+} \csc x$

Vertical Asymptotes

Definition:
The line x=a is called a **vertical asymptote** of the curve y=f(x) if either

$$\lim_{x\to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = \pm\infty$$

Example:
Basic functions we know that have VAs:

Limits at Infinity

The behaviour of functions “at” infinity is also known as the **end behaviour** or **long-term behaviour** of the function.

What happens to the y-values of a function f(x) as the x-values increase or decrease without bounds?

$$\lim_{x \to -\infty} f(x) = ?$$

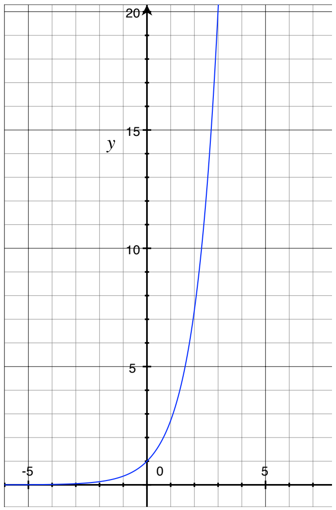
$$\lim_{x \to \infty} f(x) = ?$$

Limits at Infinity

Possibility:
y-values also approach infinity or - infinity

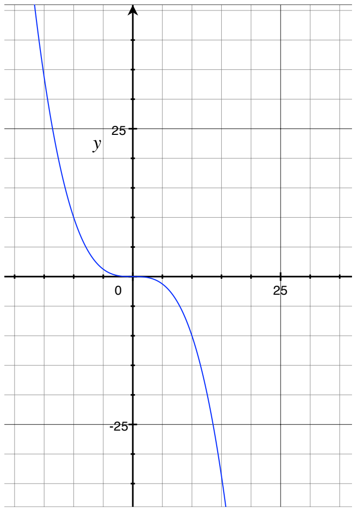
Examples:

$$f(x) = e^x$$



$$\lim_{x \to \infty} e^x = \infty \quad \text{(limit D.N.E)}$$

$$f(x) = -0.01x^3$$



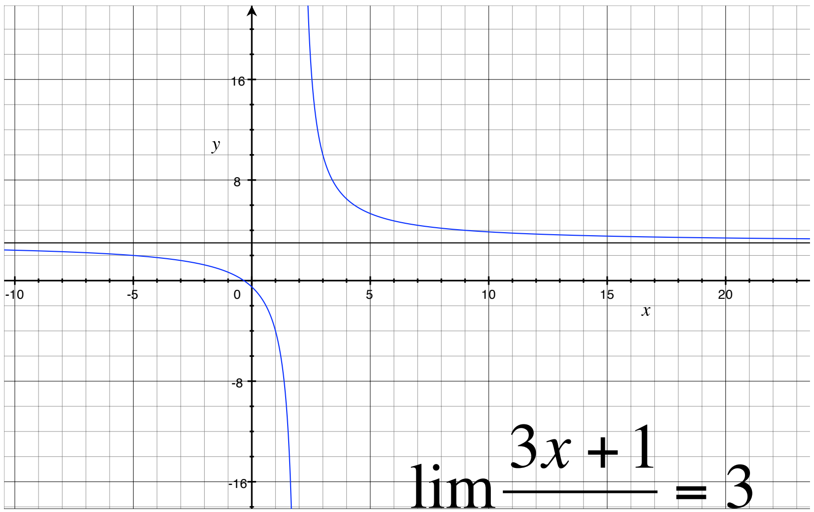
$$\lim_{x \to \infty} -0.01x^3 = -\infty \quad \text{(limit D.N.E)}$$

Limits at Infinity

Possibility:
y-values approach a unique real number L

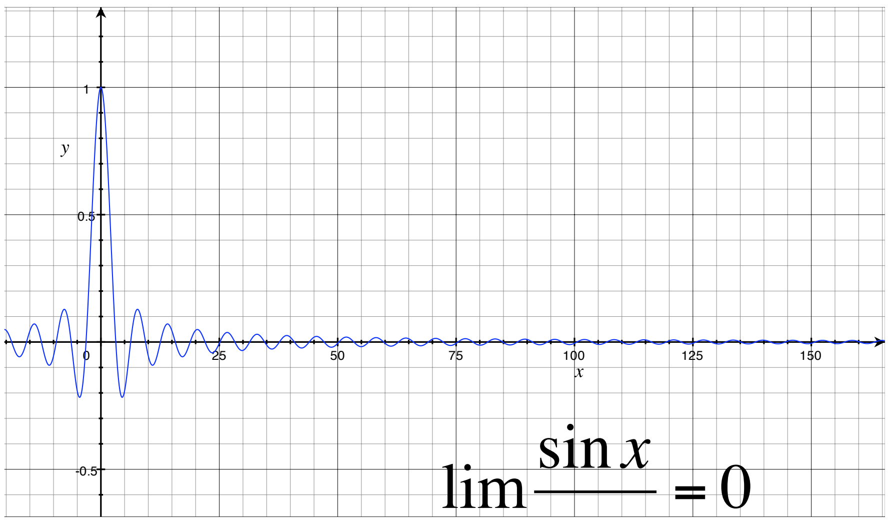
Examples:

$$f(x) = \frac{3x+1}{x-2}$$



$$\lim_{x \to \infty} \frac{3x+1}{x-2} = 3$$

$$f(x) = \frac{\sin x}{x}$$



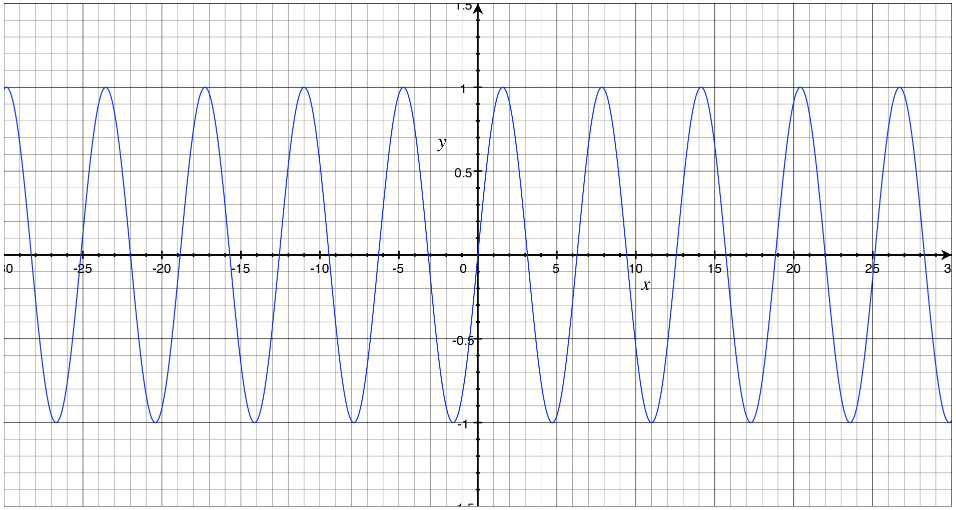
$$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

Limits at Infinity

Possibility:
y-values oscillate and do not approach a single value

Example:

$f(x) = \sin x$



$$\lim_{x \to \infty} \sin x \ \text{ D.N.E.}$$

Limits at Infinity

Definition:

$$\lim_{x \to \infty} f(x) = L$$

"the limit of f(x), as x approaches ∞, is L"

means that the values of f(x) (y-values) can be made as close as we'd like to L by taking x sufficiently large.

Definition:

$$\lim_{x \to -\infty} f(x) = L$$

"the limit of f(x), as x approaches $-\infty$, is L"

means that the values of f(x) (y-values) can be made as close as we'd like to L by taking x sufficiently small (i.e., large negative).

Calculating Limits at Infinity

*The Limit Laws listed previously are still valid if "$x \to a$" is replaced by "$x \to \infty$"

Limit Laws for Infinite Limits (abbreviated):

$$\infty + \infty = \infty$$

$$\infty \cdot \infty = \infty$$

$$c \cdot \infty = \infty$$

where c>0 is any non-zero constant

Calculating Limits at Infinity

Theorem:

If r>0 is a rational number, then $\lim_{x \to \infty} \frac{1}{x^r} = 0.$

If r>0 is a rational number such that x^r is defined for all x, then $\lim_{x \to -\infty} \frac{1}{x^r} = 0.$

Calculating Limits at Infinity

Examples:

Find the limit or show that it does not exist.

(a) $\lim_{x \to \infty} \frac{2x+1}{4-\sqrt{x}}$

(b) $\lim_{t \to \infty} 4.42857\left(\frac{\pi}{2} - \arctan\frac{2007-t}{42}\right)$

Horizontal Asymptotes

Definition:
The line y=L is called a **horizontal asymptote** of the curve y=f(x) if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

Example:
Basic functions we know that have HAs:

Limits at Infinity

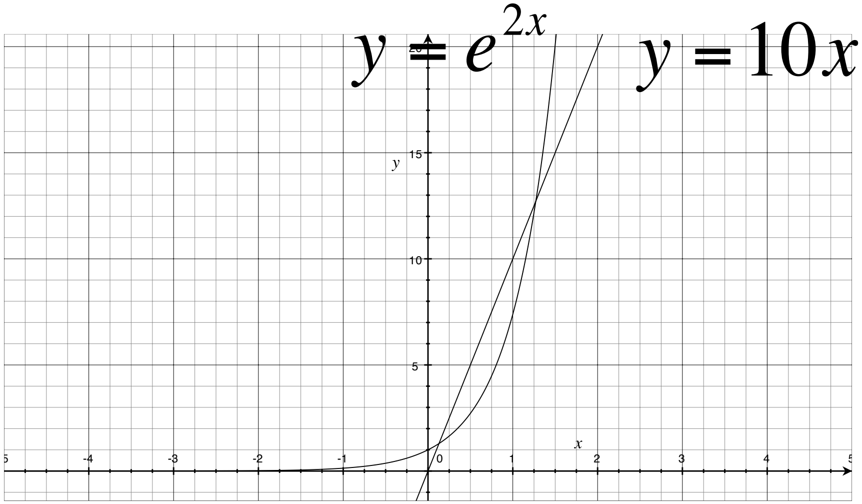
What about the limits at infinity of these functions?

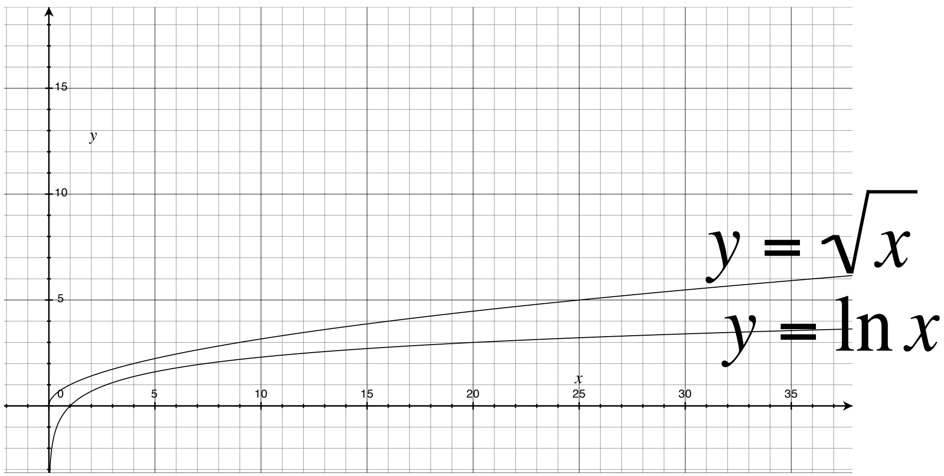
(a) $f(x) = \dfrac{e^{2x}}{10x}$

(b) $g(x) = \dfrac{\ln x}{\sqrt{x}}$

Which part (top or bottom) goes to infinity faster?

Limits at Infinity





Comparing Functions That Approach ∞ at ∞

Suppose $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$

1. f(x) approaches infinity **faster** than g(x) if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$.

2. f(x) approaches infinity **slower** than g(x) if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

3. f(x) and g(x) approach infinity at the **same rate** if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.

   where L is any finite number other than 0.

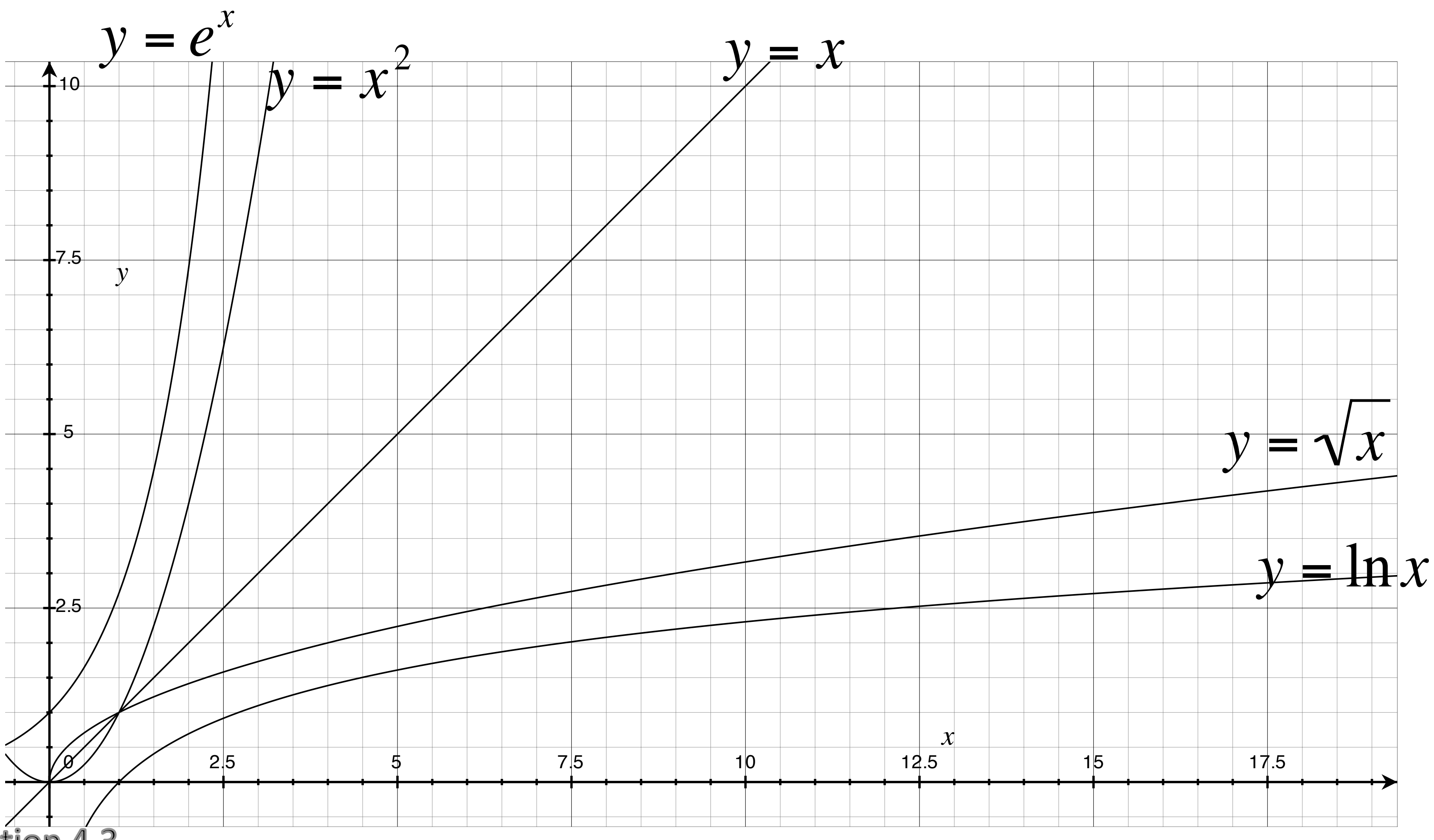
Comparing Functions That Approach ∞ at ∞

The Basic Functions in Increasing Order of Speed

Function	Comments
$a\ln x$	Goes to infinity slowly
ax^n with $n > 0$	Approaches infinity faster for larger n
$ae^{\beta x}$ with $\beta > 0$	Approaches infinity faster for larger β

Note: The constant a can be any positive number and does not change the order of the functions.

Comparing Functions That Approach ∞ at ∞



Limits at Infinity

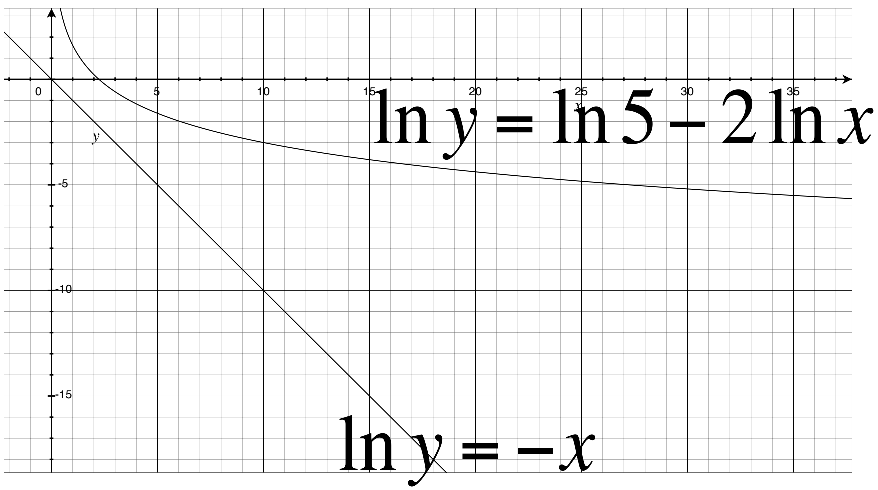
What about the limits at infinity of these functions?

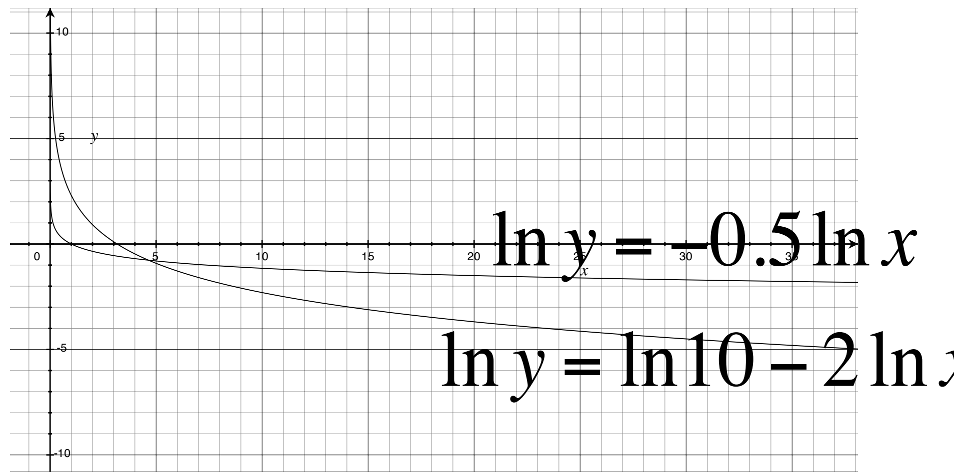
(a) $f(x) = \dfrac{e^{-x}}{5x^{-2}}$ (b) $g(x) = \dfrac{x^{-0.5}}{10x^{-2}}$

Which part (top or bottom) goes to 0 faster?

Limits at Infinity

Semilog Graphs





Comparing Functions That Approach 0 at ∞

Suppose $\lim_{x \to \infty} f(x) = 0$ and $\lim_{x \to \infty} g(x) = 0.$

1. f(x) approaches 0 **faster** than g(x) if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$

2. f(x) approaches 0 **slower** than g(x) if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty.$

3. f(x) and g(x) approach 0 at the **same rate** if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L.$

   where L is any finite number other than 0.

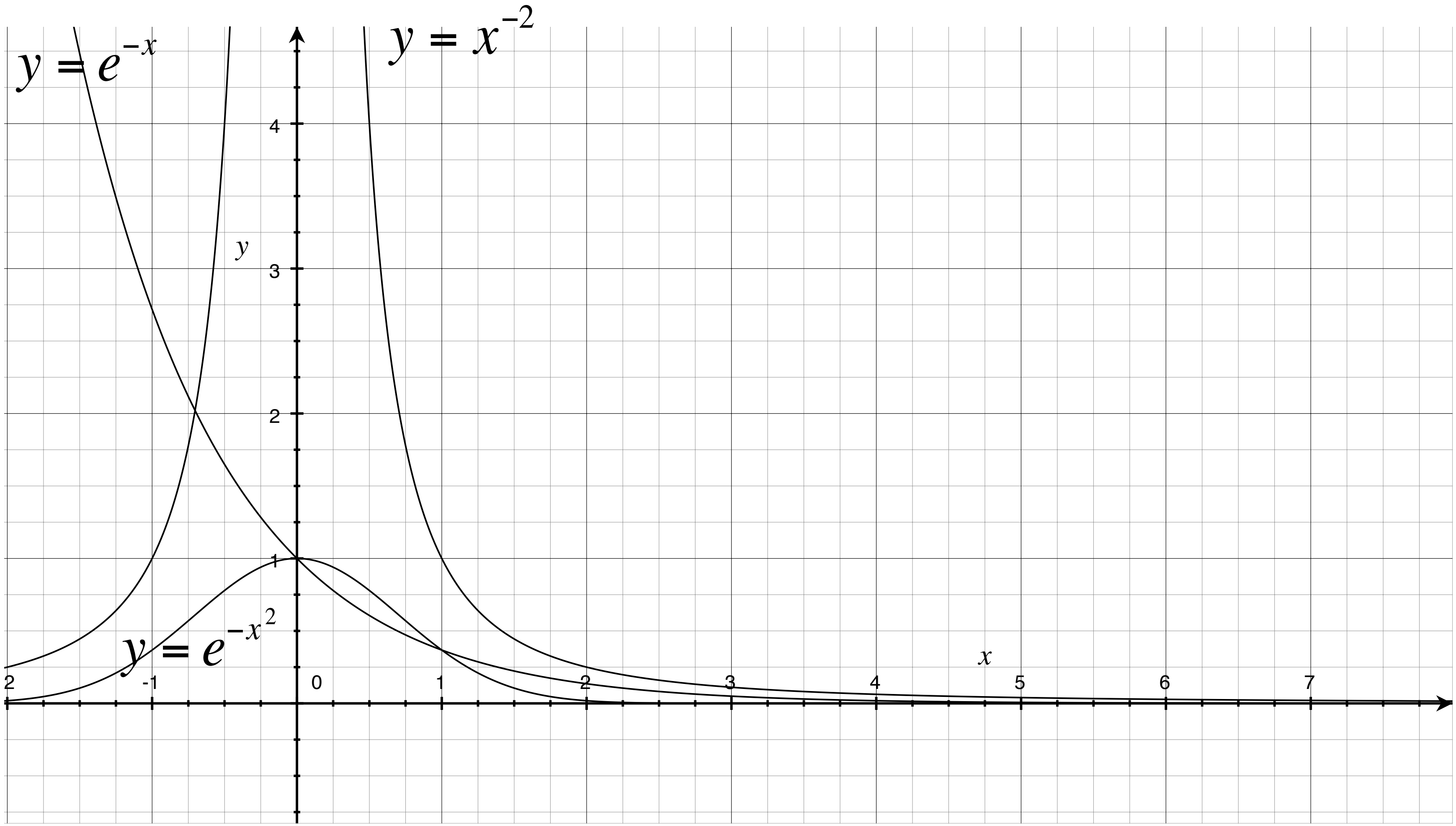
Comparing Functions That Approach 0 at ∞

The Basic Functions in Increasing Order of Speed

Function	Comments
ax^{-n} with $n > 0$	Approaches 0 faster for larger n
$ae^{-\beta x}$ with $\beta > 0$	Approaches 0 faster for larger β
$ae^{-\beta x^2}$ with $\beta > 0$	Approaches 0 really fast

Note: Again, a can be any positive constant and this will not affect the ordering.

Comparing Functions That Approach 0 at ∞



Comparing Functions That Approach 0 at ∞