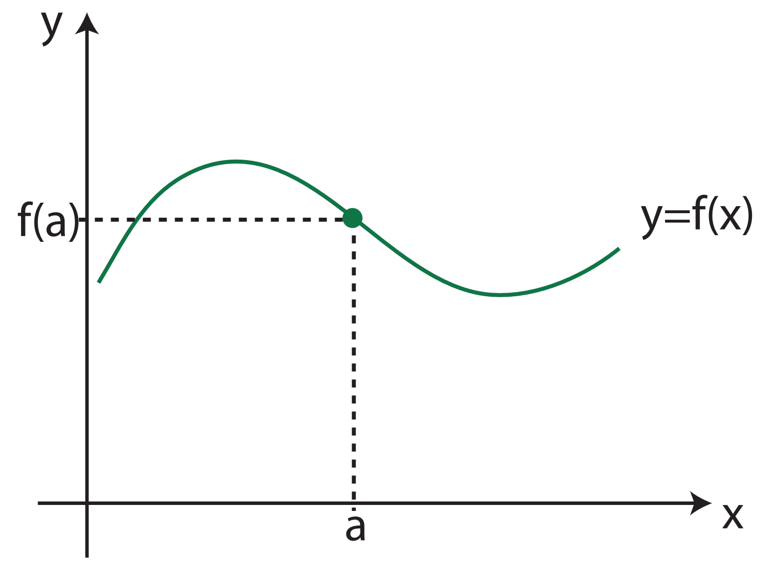
Continuity

A function is continuous at a number *a* if its behaviour *near a* (expressed in terms of the limit) matches the behaviour *at a.*



Geometrically, a function is continuous if its graph is "connected", i.e., it has no jumps or gaps.

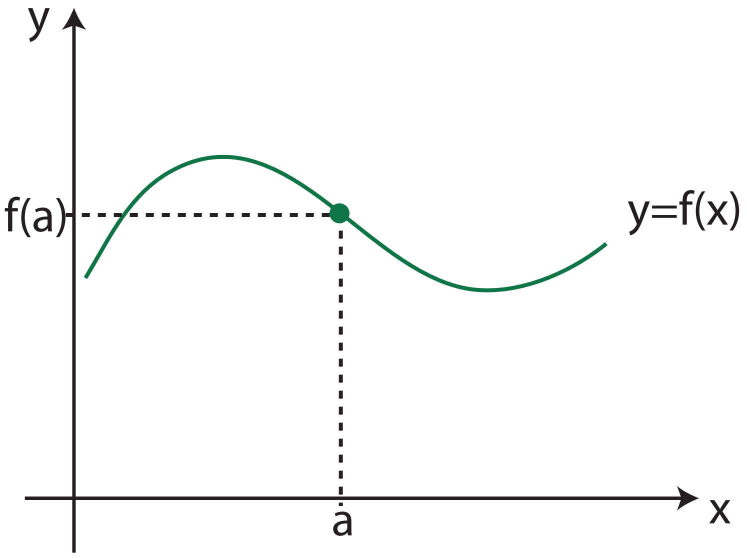
Continuity

Definition:

A function *f* is **continuous** at a number *a* if

$$\lim_{x \to a} f(x) = f(a)$$

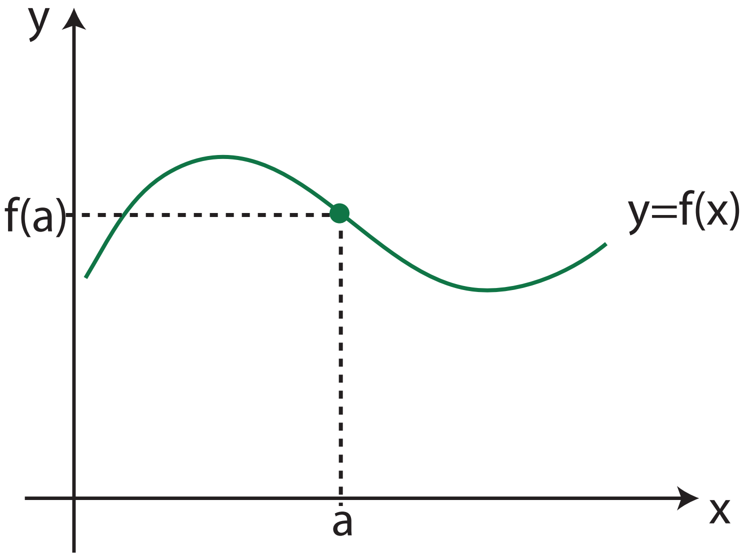
Otherwise, we say that the function is **discontinuous** at *a*.



Continuity

Implicitly requires 3 things:

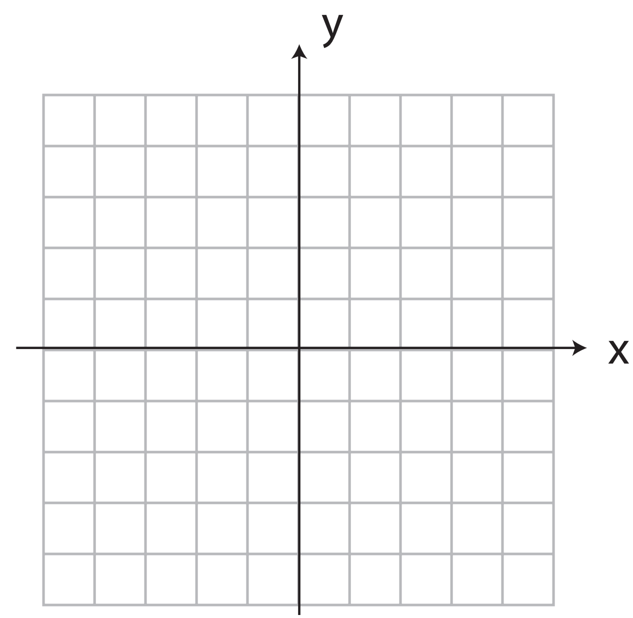
1. $\lim_{x \to a} f(x)$ is a real number
2. $f(x)$ is defined at *a*
3. $\lim_{x \to a} f(x) = f(a)$



Continuity

Example:
Find the discontinuities of the function and explain why it is discontinuous there.

$$h(x) = \frac{x+1}{x^2 - 2x - 3}$$

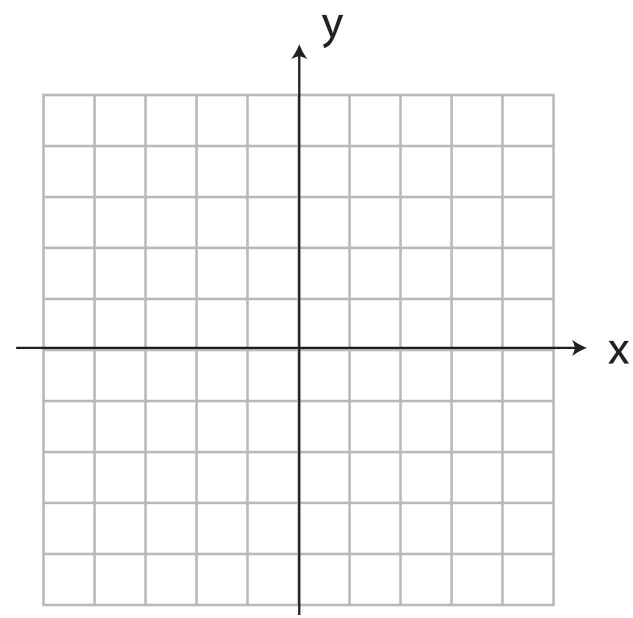


Start by looking at x-values where f(x) is **not defined** and then check the 3 conditions of continuity.

Continuity

Example:
Find the discontinuities of the function and explain why it is discontinuous there.

$$g(x) = \begin{cases} 2 - x & x \le 1 \\ \sqrt{x-1} & x > 1 \end{cases}$$



Start by looking at x-values where f(x) is **changes** from one 'piece' to another and then check the 3 conditions of continuity.

Which Functions Are Continuous?

Definition:
A function is **continuous on an interval** if it is continuous at every number in the interval.

Continuous Functions:

✓ polynomials

ex: $f(x) = 4$ $\qquad h(x) = -x^7 + 2x^4 - 1$

✓ rational functions

ex: $f(x) = \dfrac{3x-4}{1-x},\ x \neq 1$ $\qquad h(x) = \dfrac{5}{1+x^2},\ x \in R$

✓ root functions

ex: $f(x) = \sqrt{x},\ x \geq 0$ $\qquad g(x) = \sqrt[3]{x},\ x \in R$

Which Functions Are Continuous?

Continuous Functions:

✓ algebraic functions

ex: $f(x) = \dfrac{\sqrt{x+16}}{x^2+1}$

✓ absolute value function

$f(x) = |x|$

✓ exponential and logarithmic functions

ex: $f(x) = e^x$ $g(x) = \log_5 x,\ \ x > 0$

✓ trigonometric and inverse trigonometric functions

ex: $f(x) = \sin x$ $g(x) = \arctan x$

Which Functions Are Continuous?

Combining Continuous Functions:
The sum, difference, product, quotient, and composition of continuous functions is continuous where defined.

Example:
Determine where $h(x) = \dfrac{\arctan(e^x + x^2)}{\sqrt{x+1}}$ is continuous.

Which Functions Are Continuous?

By the definition of continuity, if a function *f* is continuous at *x*=*a*, then we can evaluate the limit simply by direct substitution, i.e., if *f* is continuous at *a*, then $\lim_{x \to a} f(x) = f(a).$

Example:
Evaluate $\lim_{x \to 0} \dfrac{\arctan(e^x + x^2)}{\sqrt{x+1}}$.

Interchanging a Limit and a Continuous Function

Theorem:

Assume that a function *g* satisfies $\lim_{x \to a} g(x) = b$

and that a function *f* is continuous at *b*.

Then $\lim_{x \to a} f(g(x)) = f(b)$.

Example:

Evaluate $\lim_{x \to \infty} e^{1+3x-x^2}$.