Approximating Functions with Polynomials

Polynomials have many nice properties and are generally very easy to work with.

For this reason, it is often useful to approximate more complicated functions with polynomials in order to simplify calculations.

Linear functions are the simplest polynomials and can be used to represent a more complicated function in many situations.

Linear Approximations

The **secant line** connecting points (a, $f(a)$) and (b, $f(b)$) on the graph of $f(x)$ provides a decent linear approximation to $f(x)$ for x-values in the interval [a, b].

The **tangent line** to the graph of $f(x)$ at (a, $f(a)$) provides the best linear (straight line) approximation to $f(x)$ *near* $x=a$.

Linear (or tangent line) approximation to $f(x)$ around $x=a$:

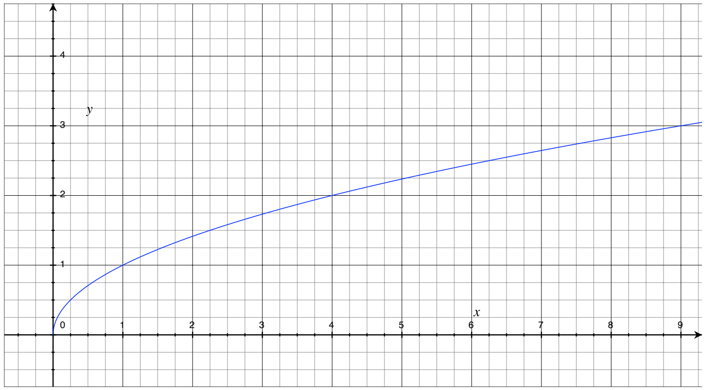
$$L(x) = f(a) + f'(a)(x - a)$$

Linear Approximations

Example:
Let $f(x) = \sqrt{x}$.

Find the secant line approximation to *f*(*x*) for *x*-values between 1 and 4.

Use this to approximate both $\sqrt{2}$ and $\sqrt{3}$.



__Using Technology__:

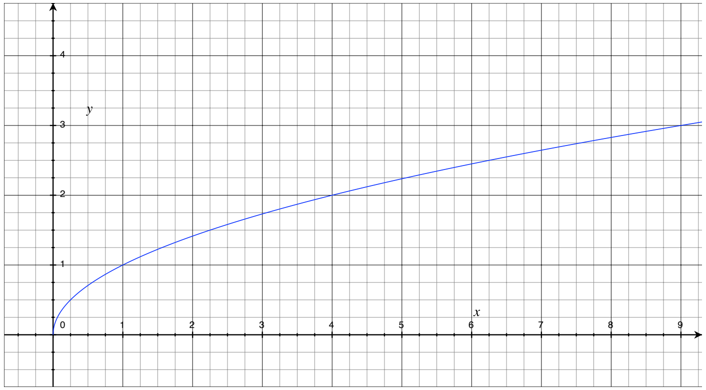
$\sqrt{2} \approx 1.41421356237$
$\sqrt{3} \approx 1.73205080757$

Linear Approximations

Example:

Let $f(x) = \sqrt{x}$.

Approximate the values of $\sqrt{2}$ and $\sqrt{3}$ using suitable tangent line approximations.



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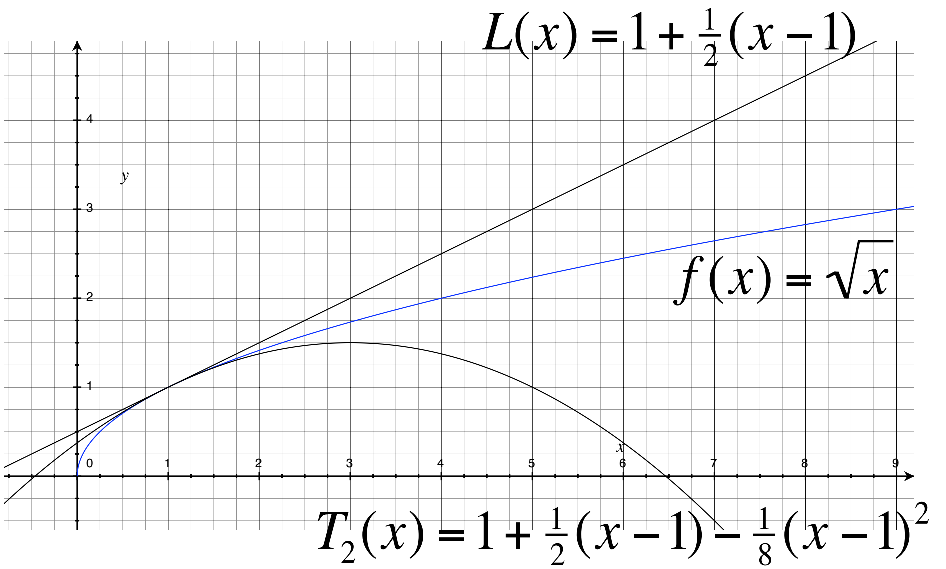
Quadratic Approximations

We can obtain a more accurate approximation by using polynomials of higher degrees.

The ***tangent line (linear) approximation*** matches the value and the slope of the function at x=1.

The ***quadratic approximation*** matches the value, the slope, and the curvature (concavity) of the function at x=1.



Quadratic Approximations

To find the quadratic approximation to a function $f(x)$ at the base point *a*, we match the value, the first derivative, and the second derivative at the point *a*.

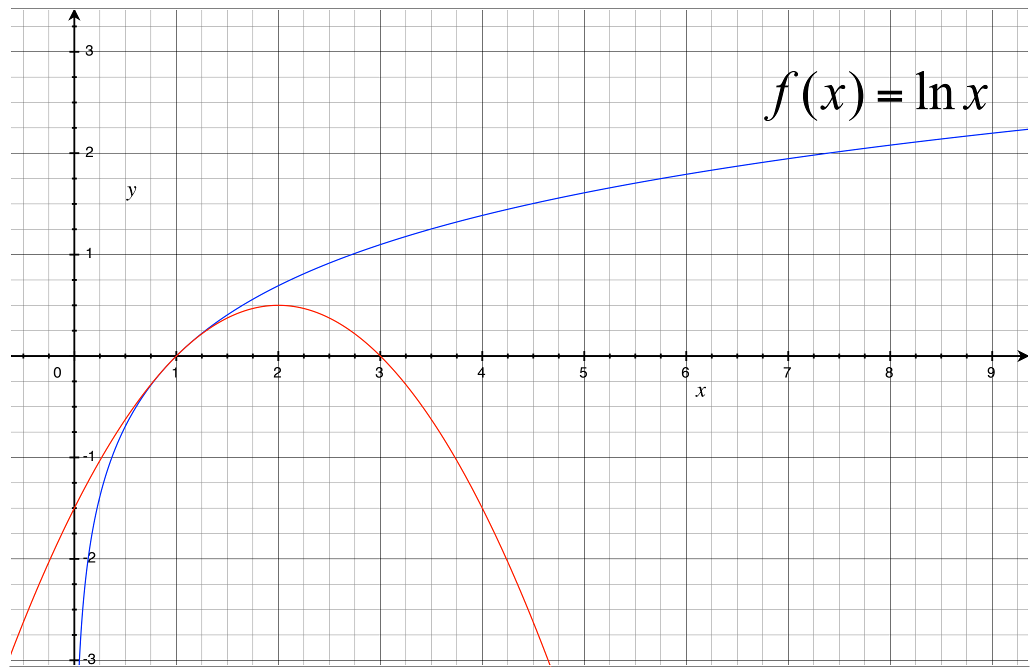
<u>Quadratic approximation to $f(x)$ around *x=a*</u>:

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

because the
degree is 2

Quadratic Approximations

Example:
Estimate the value of $\ln 1.1$ using a quadratic approximation and a suitable base point.



The Taylor Polynomial

Suppose the first *n* derivatives of the function *f* are defined at $x=a$.
Then the **Taylor polynomial of degree *n*** matching the values of the first *n* derivatives is

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

n^{th} derivative

$$+ \frac{f^{(i)}(a)}{i!}(x-a)^i + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Taylor Polynomial

Example:

Find the 5th degree Taylor polynomial for $f(x) = \sin x$ near $x=0$.

The Taylor Polynomial