Differential Equations

A **differential equation** is an equation that involves an unknown function and one or more of its derivatives.

Examples:

$$y' = x^2 + e^x \qquad y' = 2 + y \qquad y' = x + y$$

Differential Equations

A **solution** of a differential equation is a function that, along with its derivatives, satisfies the DE.

Example:

Show that $y = 2 + e^{-x^3}$ is a solution of the differential equation $y' + 3x^2 y = 6x^2$.

Before		
Value (measured)	differentiate →	Rate Of Change (computed)
position, s(t)		velocity, s′(t)
population, P(t)		growth rate, P′(t)

Now		
Rate of Change (measured)	solve differential equation →	Value (computed)
velocity, s′(t)		position, s(t)
growth rate, P′(t)		population, P(t)

Pure-Time DEs

A pure-time differential equation is obtained by ***measuring*** the rate of change of the unknown quantity and expressing it as a function of time.

Example: $$\frac{ds}{dt} = t^2 - 3t + 5$$

** Note that the formula for the rate of change depends ***purely*** on the ***time*** *t*.

Example 1: Volume of a Cell

Suppose we observe that $2.0\ \mu m^3$ of water enters a cell each second.

Differential Equation:

General Solution:

** V is called the 'state variable'

Example 1: Volume of a Cell

Suppose we are told that the initial volume of the cell is $150\,\mu m^3$.

General Solution:

Initial Condition:

Particular Solution:

Autonomous DEs

An autonomous differential equation is derived from a ***rule*** describing how a quantity changes and is expressed as a function of the unknown quantity.

Example: $$\frac{dS}{dt} = \alpha S$$

This expresses the fact that the quantity *S(t)* grows at a rate proportional to its size.

Example 2: Population Size

Suppose we know that a quantity *P(t)* is initially P_0 and grows at a rate proportional to its size.

Differential Equation:

Initial Condition:

Particular Solution:

Modelling: Verbal Descriptions ➡ IVPs

Example:

It has been observed that the relative rate of change of a population of wild foxes in an ecosystem is 0.75 baby foxes per fox per month. Initially, the population is 74 thousand.

Write a differential equation and an initial condition to describe the event. Then, solve the initial value problem.

Solutions for General DEs

- **Algebraic Solutions**
  - an explicit formula or algorithm for the solution (often, impossible to find)
- **Geometric Solutions**
  - a sketch of the solution obtained from analyzing the DE
- **Numeric Solutions**
  - an approximation of the solution using technology and and some estimation method, such as Euler's method

Graphical Solutions of Pure-Time DEs

Example:

Sketch the graph of the solution to $s'(t) = \ln t$ given the initial condition $s(1) = 1$.

Euler's Method

What information does an initial value problem tell us about the solution?

Example:

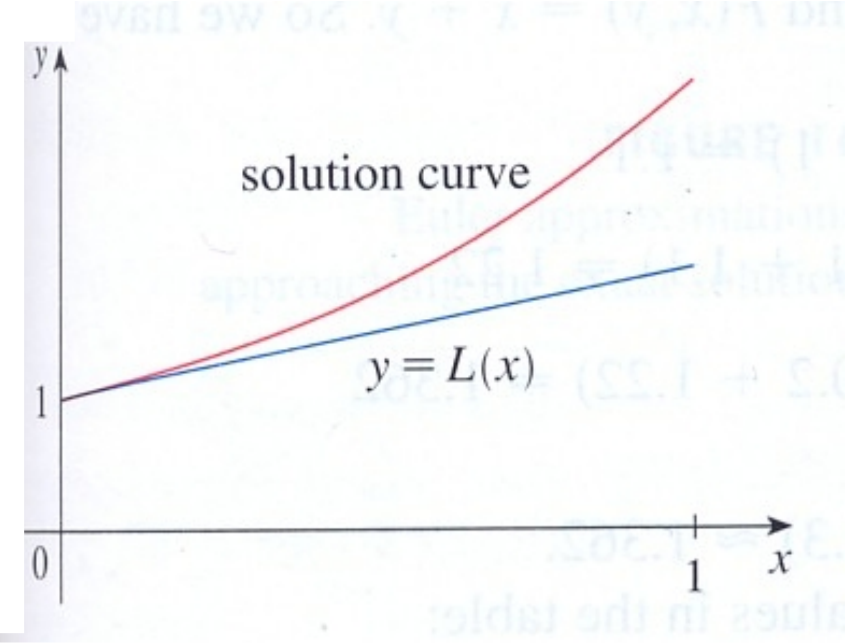
DE: $\dfrac{dy}{dx} = x + y$ ← slope of the solution curve y(x)

IC: $y(0) = 1$ ← an exact value of the solution

Euler's Method

Euler's Idea:

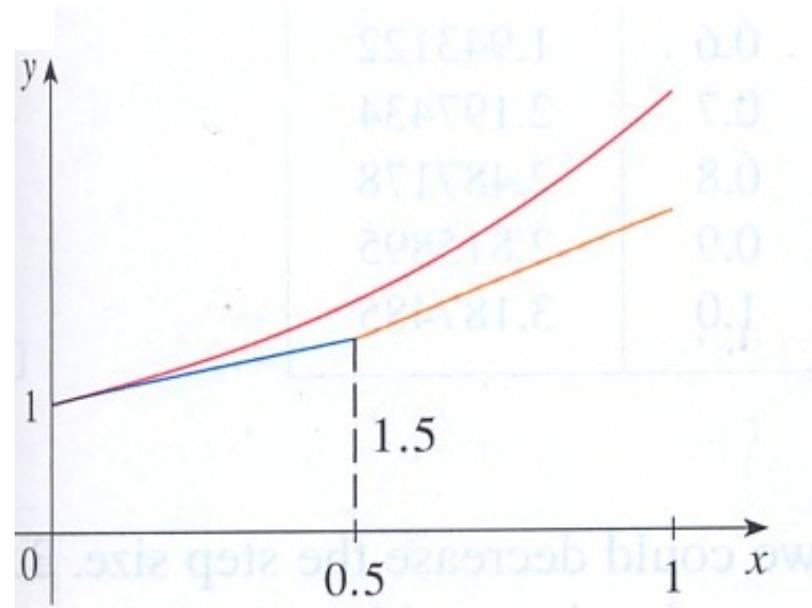
First, using the initial condition as a base point, approximate the solution curve y(x) by its tangent line.



First Euler approximation

Euler’s Method

Next, travel a short distance along this line, determine the slope at the new location (using the DE), and then proceed in that ‘corrected’ direction.

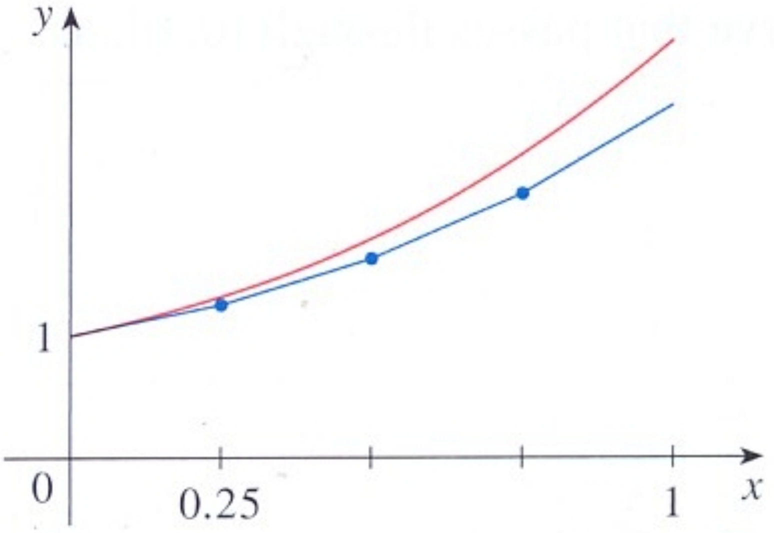


Euler’s approximation with step size $\Delta x = 0.5$

Euler's Method

Repeat, correcting your direction midcourse using the DE at regular intervals to obtain an approximate solution of the IVP.

By increasing the number of midcourse corrections, we can improve our estimation of the solution.



Euler approximation with step size $\Delta x = 0.25$

Euler's Method

Summary:

An approximate solution to the IVP

$$\frac{dy}{dt} = G(t,y), \quad y(t_0) = y_0$$

is generated by choosing a step size Δt and computing values according to the algorithm

$$t_{n+1} = t_n + \Delta t$$

$$y_{n+1} = y_n + G(t_n, y_n)\Delta t$$

Euler's Method

Algorithm:

$$t_{n+1} = t_n + \Delta t$$
$$y_{n+1} = y_n + G(t_n, y_n)\Delta t$$

Algorithm In Words:

next time = current time + step size

next approximation = current approximation + rate of change at current values x step size

Example

Consider the IVP

$$y' = x + y \quad y(0) = 1$$

Approximate the value of the solution at x=1 by applying Euler's method and using a step size of 0.25.

Example

Calculations:

Table of Approximate Values for the Solution y(x) of the IVP

t_n	y_n
$x_0 = 0$	$y_0 = 1$

Qualitative Analysis of a DE

Example:

A population of caribou is modeled by

$$\frac{dP}{dt} = 2P(t)\left(1 - \frac{P(t)}{2500}\right)\left(1 - \frac{120}{P(t)}\right), \;\; P(t) > 0.$$

In which of the following situations will the population increase in the immediate future?

(I) P(0)=100 (II) P(0)=200 (III) P(0)=3000