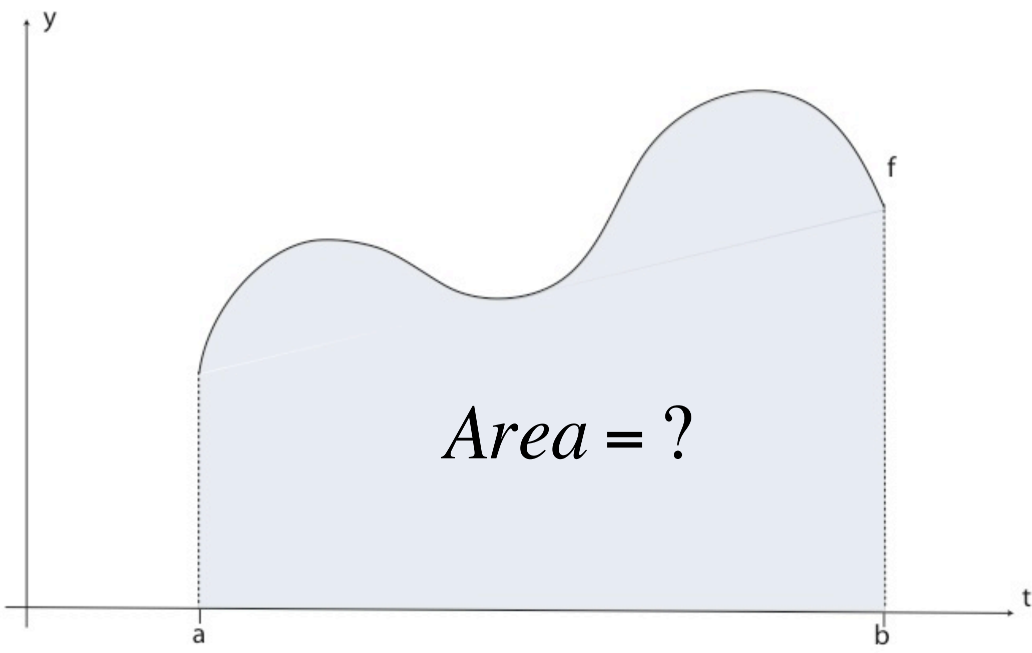
Area

How do we calculate the area of some irregular shape?

For example, how do we calculate the area under the graph of f on [a,b]?



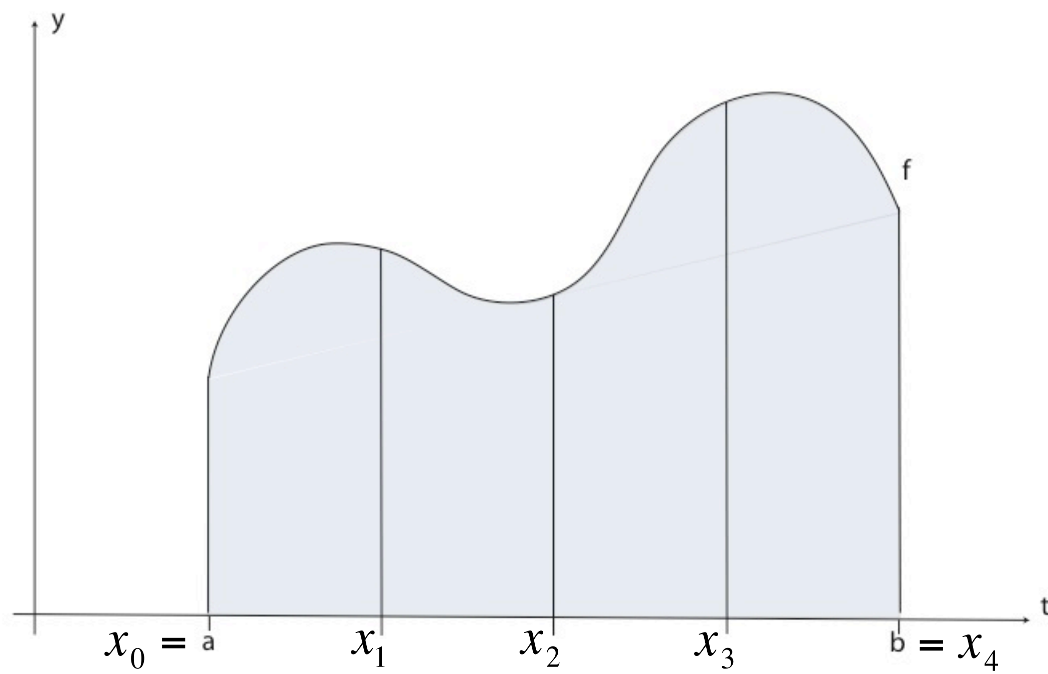
Area

Approach:

We approximate the area using rectangles.

number of rectangles: $n = 4$

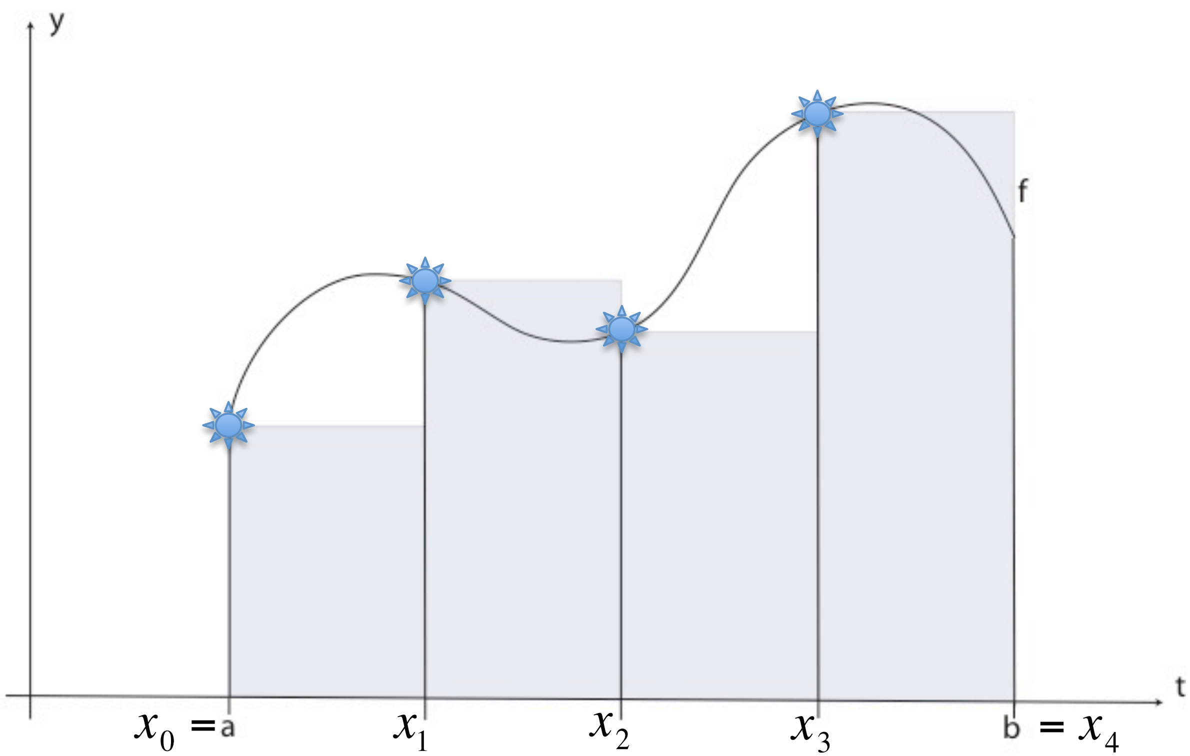
width of each rectangle: $\Delta x = \dfrac{b-a}{n}$



Area

Left-hand estimate:

Let the height of each rectangle be given by the value of the function at the left endpoint of the interval.



Area

<u>*Left-hand estimate*</u>:

$$Area \approx f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$$

$$\approx (f(x_0) + f(x_1) + f(x_2) + f(x_3))\Delta x$$

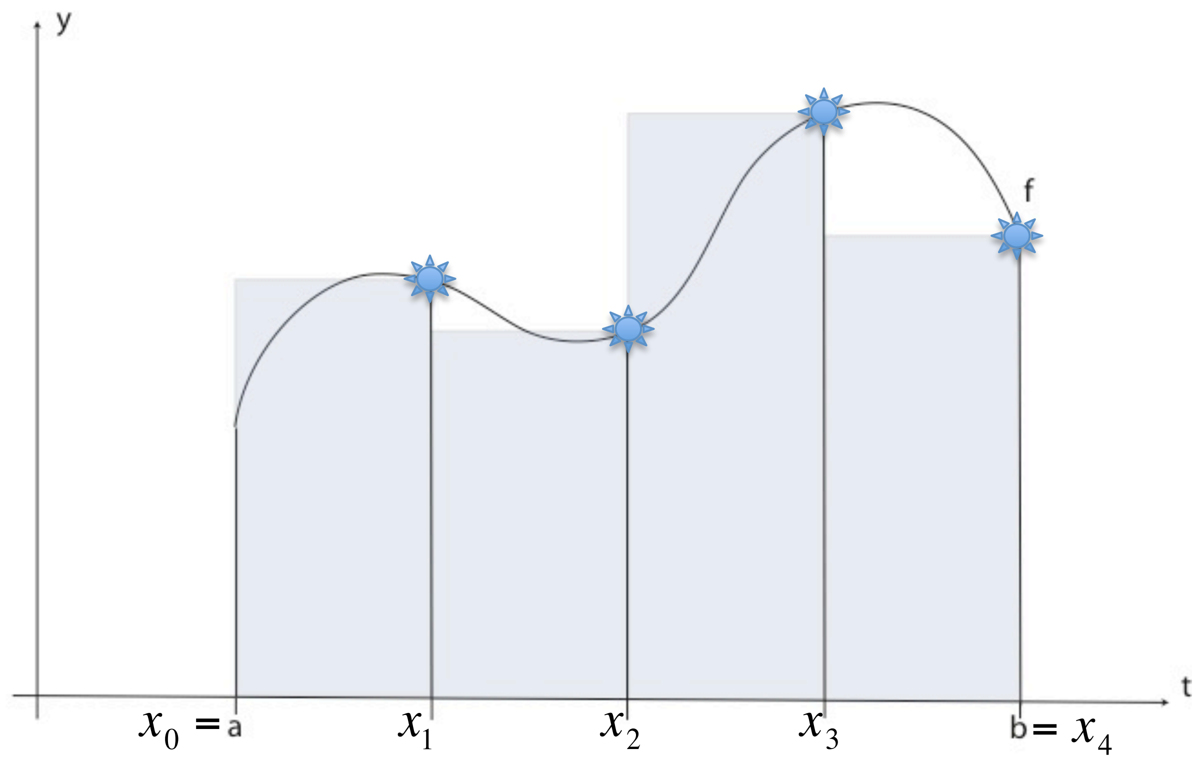
$$\approx \sum_{i=0}^{3} f(x_i)\Delta x$$

Riemann Sum

Area

Right-hand estimate:

Let the height of each rectangle be given by the value of the function at the right endpoint of the interval.



Area

<u>*Right-hand estimate*</u>:

$$Area \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$\approx (f(x_1) + f(x_2) + f(x_3) + f(x_4))\Delta x$$

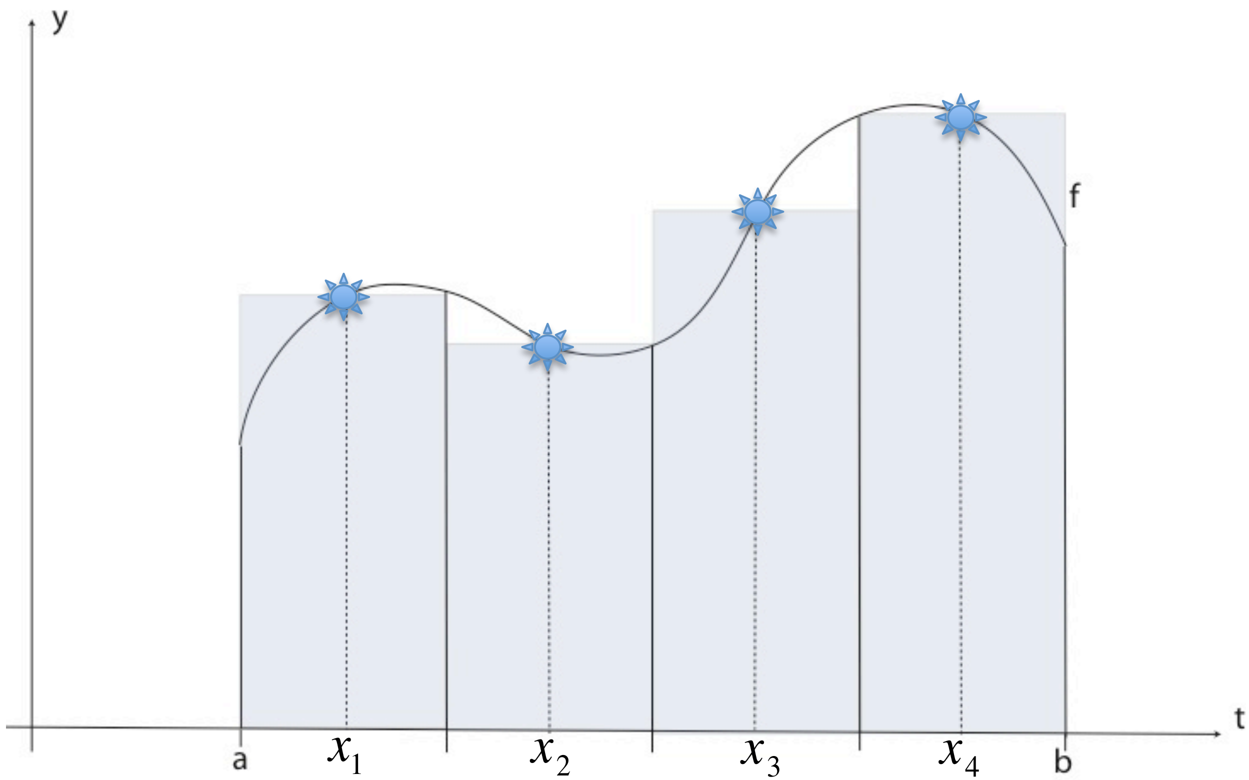
$$\approx \sum_{i=1}^{4} f(x_i)\Delta x$$

Riemann Sum

Area

<u>*Midpoint estimate*</u>:

Let the height of each rectangle be given by the value of the function at the midpoint of the interval.



Area

Midpoint estimate:

$$Area \approx f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + f(x_4^*)\Delta x$$

$$\approx (f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*))\Delta x$$

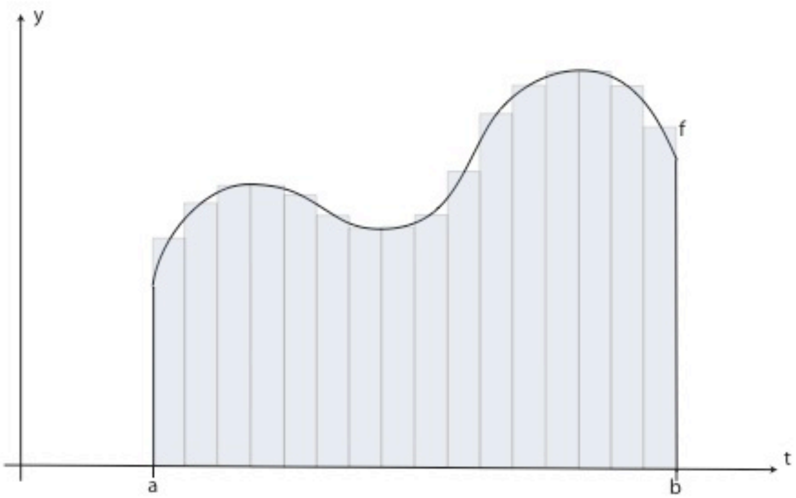
$$\approx \sum_{i=1}^{4} f(x_i^*)\Delta x$$

Riemann Sum

Area

How can we improve our estimation?

Increase the number of rectangles!!!



$$Area \approx \sum_{i=1}^{16} f(x_i^*)\Delta x$$

How do we make it exact?

Riemann Sums and the Definite Integral

__Definition__:

The **definite** integral of a function f on the interval from **a** to **b** is defined as a limit of the Riemann sum

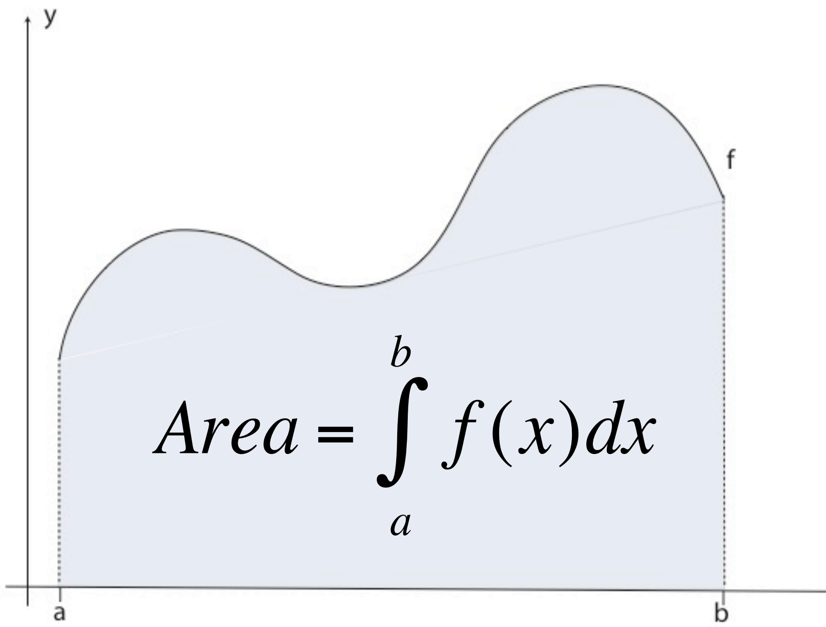
$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where x_i^* is some sample point in the interval $[x_{i-1}, x_i]$ and $\Delta x = \dfrac{b-a}{n}$.

The Definite Integral

Interpretation:

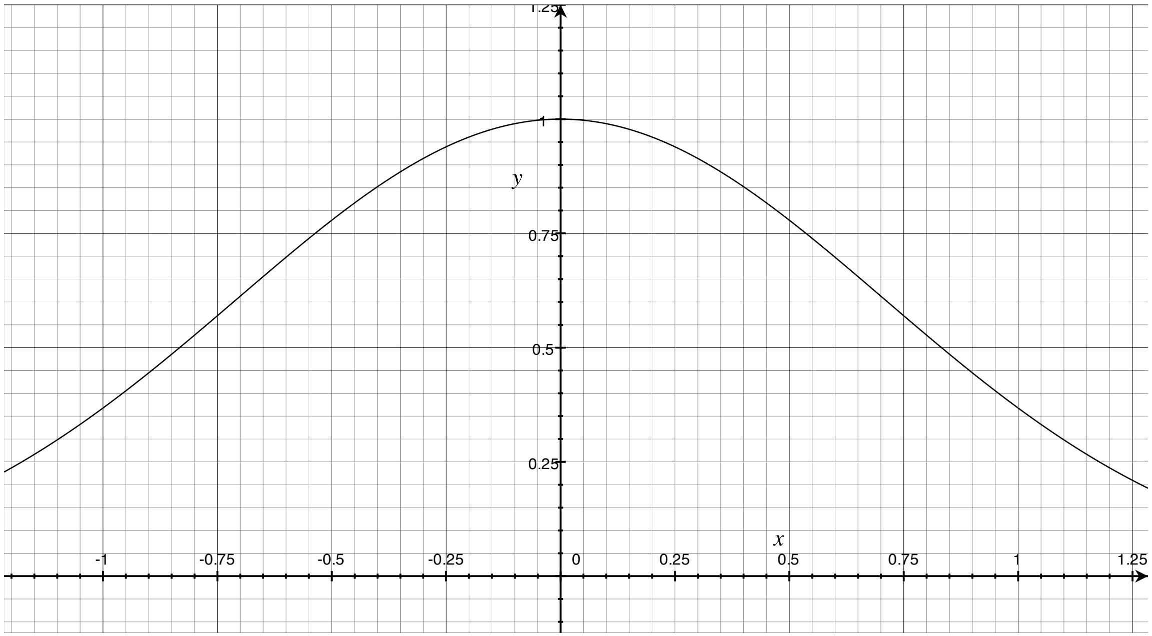
If $f \geq 0$, then the definite integral is the area under the curve $y = f(x)$ from a to b.



The Definite Integral

Example:

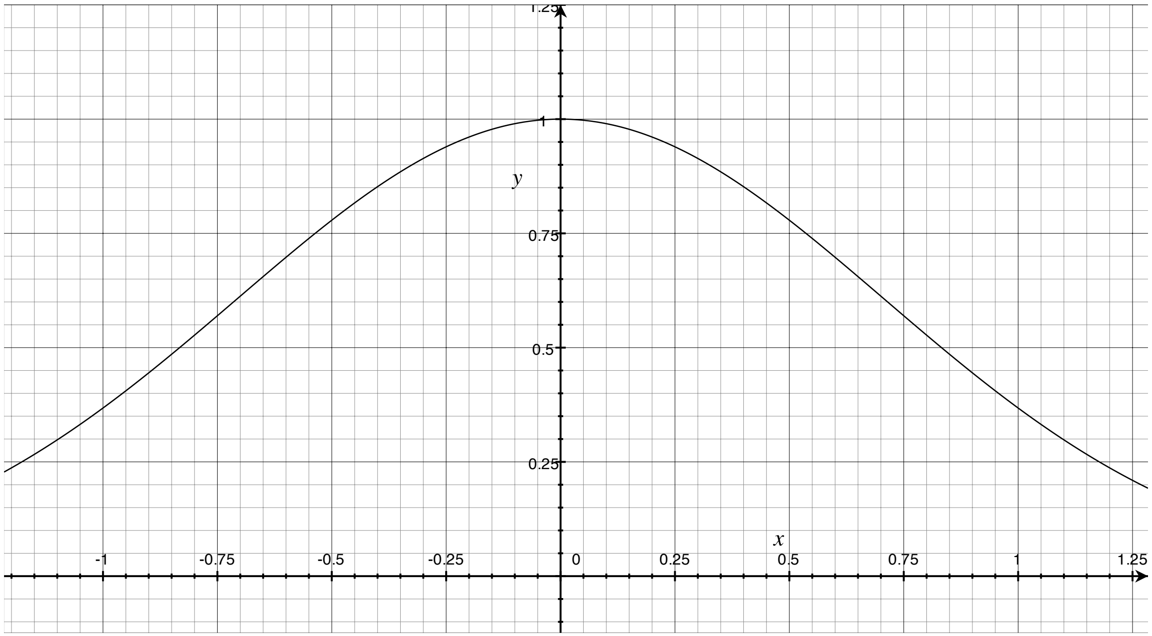
Estimate the value of $\int_0^1 e^{-x^2}\,dx$ using two rectangles and left-endpoints.



The Definite Integral

Example:

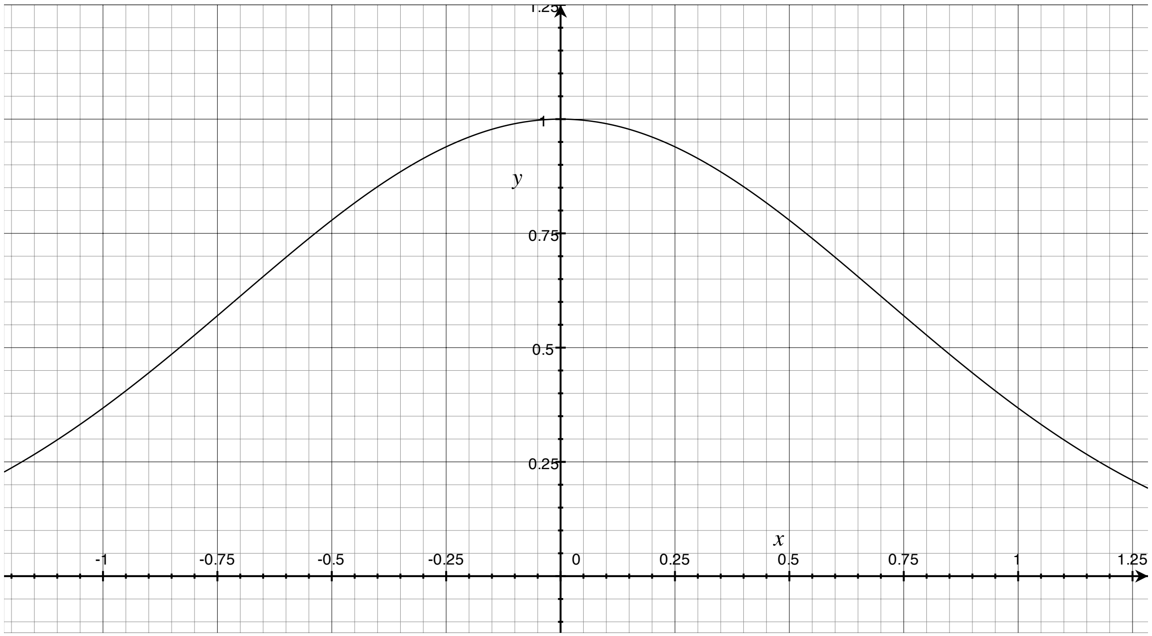
Estimate the value of $\int_0^1 e^{-x^2}\,dx$ using two rectangles and right-endpoints.



The Definite Integral

Example:

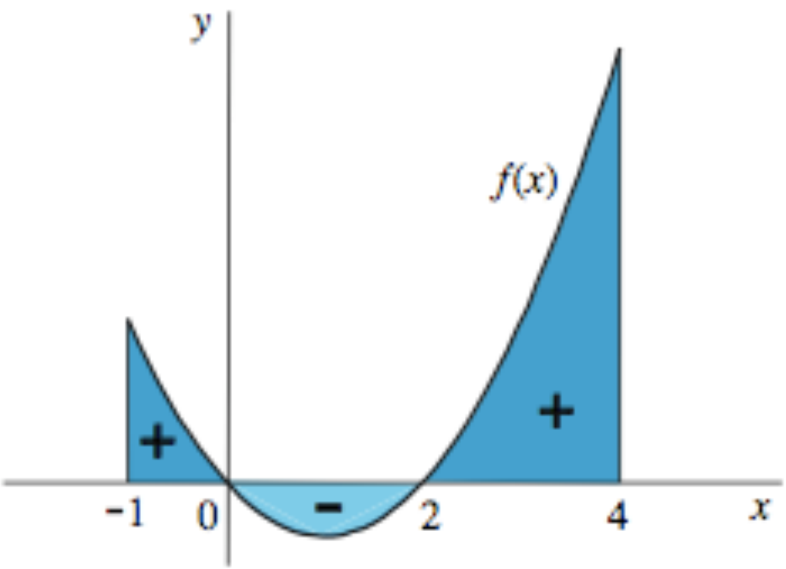
Estimate the value of $\int_0^1 e^{-x^2}\,dx$ using two rectangles and midpoints.



The Definite Integral

<u>*Interpretation*</u>:

If f is both positive and negative, then the definite integral represents the NET or SIGNED area, i.e. the area above the x-axis and below the graph of f minus the area below the x-axis and above f



$$\int_{-1}^{4} f(x)dx = net\ area$$

Definite Integrals and Area

Example:

Evaluate the following integrals by interpreting each in terms of area.

(a) $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$

(b) $\displaystyle\int_0^3 (x-1)\,dx$

(c) $\displaystyle\int_{-\pi}^{\pi} \sin x\,dx$

Properties of Integrals

Assume that f(x) and g(x) are continuous functions and a, b, and c are real numbers such that a<b.

$$(1)\ \int_a^a f(x)\,dx = 0$$

$$(2)\ \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$(3)\ \int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

Properties of Integrals

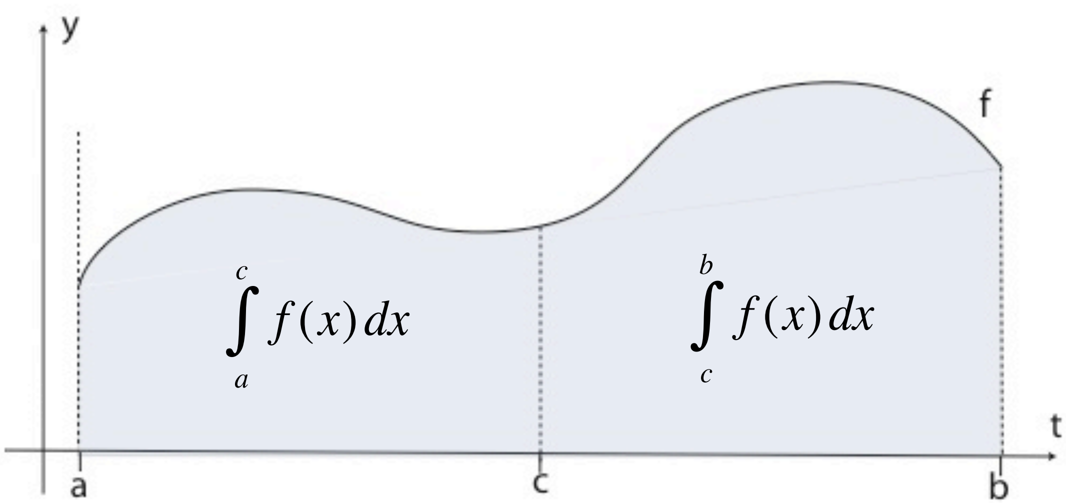
Assume that f(x) and g(x) are continuous functions and a, b, and c are real numbers such that a<b.

$$(4)\ \int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$(5)\ \int_a^b c\,dx = c(b-a)$$

Properties of Integrals

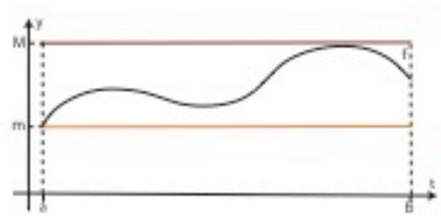
(6) Suppose f(x) is continuous on the interval from a to b and that $a \le c \le b$.

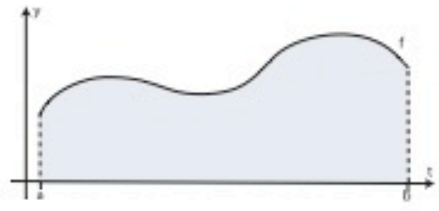


Then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

Properties of Integrals

(7) Suppose f(x) is continuous on the interval from a to b and that $m \le f(x) \le M$.





Then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.