The Chain Rule and Integration by Substitution

Recall:

The chain rule for derivatives allows us to differentiate a composition of functions:

derivative

$$[f(g(x))]' = f'(g(x))g'(x)$$

antiderivative

The Chain Rule and Integration by Substitution

Suppose we have an integral of the form

$$\int f(g(x))g'(x)dx \qquad \text{where} \qquad F' = f.$$

composition of functions

derivative of Inside function

F is an antiderivative of f

Then, by reversing the chain rule for derivatives, we have

$$\int \underbrace{f(g(x))g'(x)}_{\substack{\text{integrand is the result of} \\ \text{differentiating a composition} \\ \text{of functions}}} dx = F(g(x)) + C.$$

Example

Integrate $\int \frac{2x+5}{\sqrt{x^2+5x-7}}\,dx$

Integration by Substitution

<u>*Algorithm*</u>:

1. Let $u = g(x)$ where $g(x)$ is the part causing problems and $g'(x)$ cancels the remaining x terms in the integrand.

2. Substitute $u = g(x)$ and $du = g'(x)dx$ into the integral to obtain an equivalent (easier!) integral all in terms of u.

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Integration by Substitution

Algorithm:

3. Integrate with respect to u, if possible.

$$\int f(u)du = F(u) + C$$

4. Write final answer in terms of x again.

$$F(u) + C = F(g(x)) + C$$

Integration by Substitution

Example:

Integrate each using substitution.

(a) $\int xe^{4x^2}\,dx$

(b) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

(c) $\int_1^{e^4} \frac{\ln x}{x}\,dx$

The Product Rule and Integration by Parts

The product rule for derivatives leads to a technique of integration that breaks a complicated integral into simpler parts.

<u>Integration by Parts Formula</u>:

$$\int u\,dv = uv - \int v\,du$$

given integral that
we cannot solve

hopefully this is a simpler
Integral to evaluate

The Product Rule and Integration by Parts

Deriving the Formula

Start by writing out the Product Rule:

$$\frac{d}{dx}[u(x)\cdot v(x)] = \frac{du}{dx}\cdot v(x) + u(x)\cdot\frac{dv}{dx}$$

Solve for $u(x)\cdot\frac{dv}{dx}$:

$$u(x)\cdot\frac{dv}{dx} = \frac{d}{dx}[u(x)\cdot v(x)] - \frac{du}{dx}\cdot v(x)$$

The Product Rule and Integration by Parts

Deriving the Formula

Integrate both sides with respect to x:

$$\int u(x)\frac{dv}{dx}\,dx = \int \frac{d}{dx}[u(x)\cdot v(x)]\,dx - \int v(x)\frac{du}{dx}\,dx$$

The Product Rule and Integration by Parts

Deriving the Formula

Simplify:

$$\int u(x)\frac{dv}{\cancel{dx}}\cancel{dx} = \cancel{\int}\frac{\cancel{d}}{\cancel{dx}}[u(x)\cdot v(x)]\cancel{dx} - \int v(x)\frac{du}{\cancel{dx}}\cancel{dx}$$

$$\Rightarrow \quad \int u(x)dv = u(x)\cdot v(x) - \int v(x)du$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Template:

Choose:

$u =$ part which gets simpler after differentiation

$dv =$ easy to integrate part

Compute:

$du =$

$v =$

Integration by Parts

Example:

Integrate each using integration by parts.

(a) $\int x\cos 4x\,dx$

(b) $\int x^2 e^{\frac{x}{2}}\,dx$

(c) $\int_1^2 \ln x\,dx$

Strategy for Integration

Method	Applies when…
Basic antiderivative	…the integrand is recognized as the reversal of a differentiation formula, such as
Guess-and-check	…the integrand differs from a basic antiderivative in that "x" is replaced by "ax+b", for example
Substitution	…both a function and its derivative (up to a constant) appear in the integrand, such as
Integration by parts	…the integrand is the product of a power of x and one of sin x, cos x, and e^x, such as …the integrand contains a single function whose derivative we know, such as

Strategy for Integration

What if the integrand does not have a formula for its antiderivative?

Example:

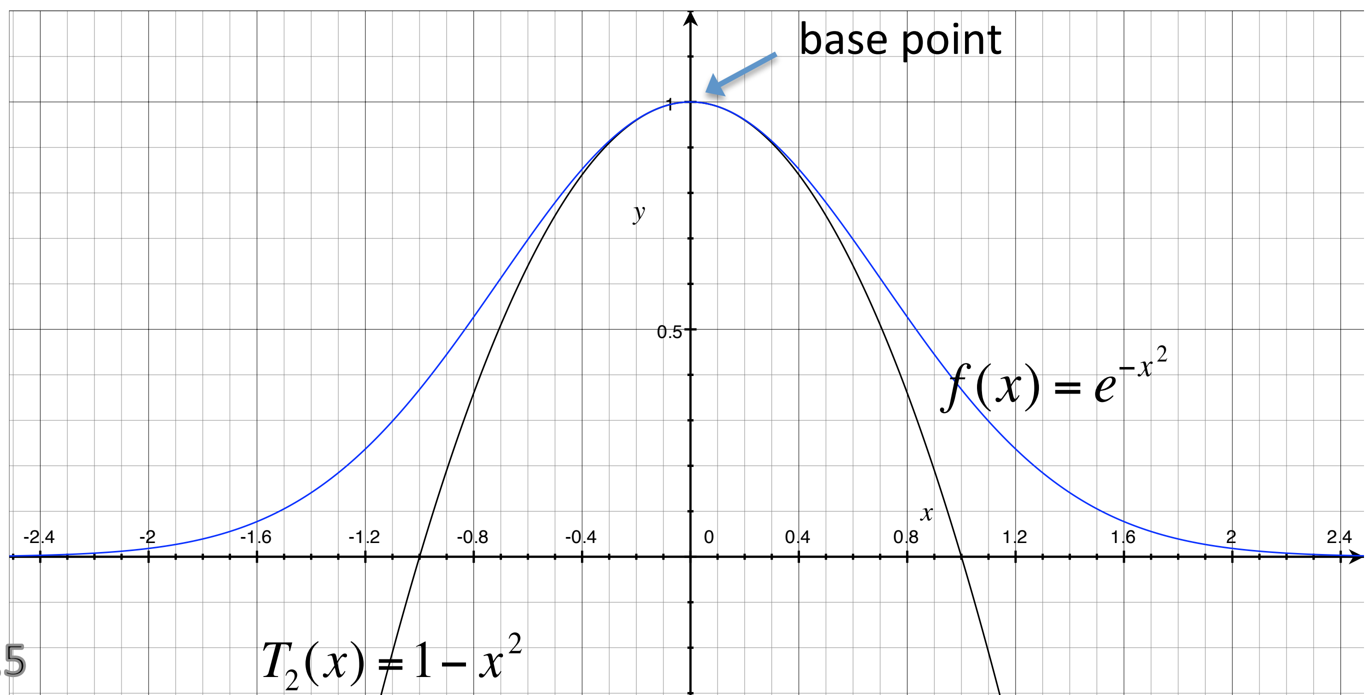
impossible to integrate

$$\int_0^1 e^{-x^2}\,dx$$

Approximating Functions with Polynomials

Recall:

The quadratic approximation to $f(x) = e^{-x^2}$ around the base point x=0 is $T_2(x) = 1 - x^2$.



Integration Using Taylor Polynomials

We approximate the function with an appropriate Taylor polynomial and then integrate this Taylor polynomial instead!

Example:

impossible to integrate

easy to integrate

$$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 (1 - x^2)\,dx$$

for x-values near 0

Integration Using Taylor Polynomials

We can obtain a better approximation by using a higher degree Taylor polynomial to represent the integrand.

Example:

$$\int_0^1 e^{-x^2}\,dx$$

$$\approx \int_0^1 \left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6\right) dx$$

$$\approx \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7\right|_0^1$$

$$\approx 0.74286$$