The Definite Integral – Area Between Curves

The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is

$$A = \int_a^b \left| f(x) - g(x) \right| dx$$

<u>Recall</u>:

$$\left| f(x) - g(x) \right| = \begin{cases} f(x) - g(x) & when \quad f(x) \geq g(x) \\ g(x) - f(x) & when \quad f(x) \leq g(x) \end{cases}$$

The Definite Integral – Area Between Curves

Examples:

Sketch the region enclosed by the given curves and then find the area of the region.

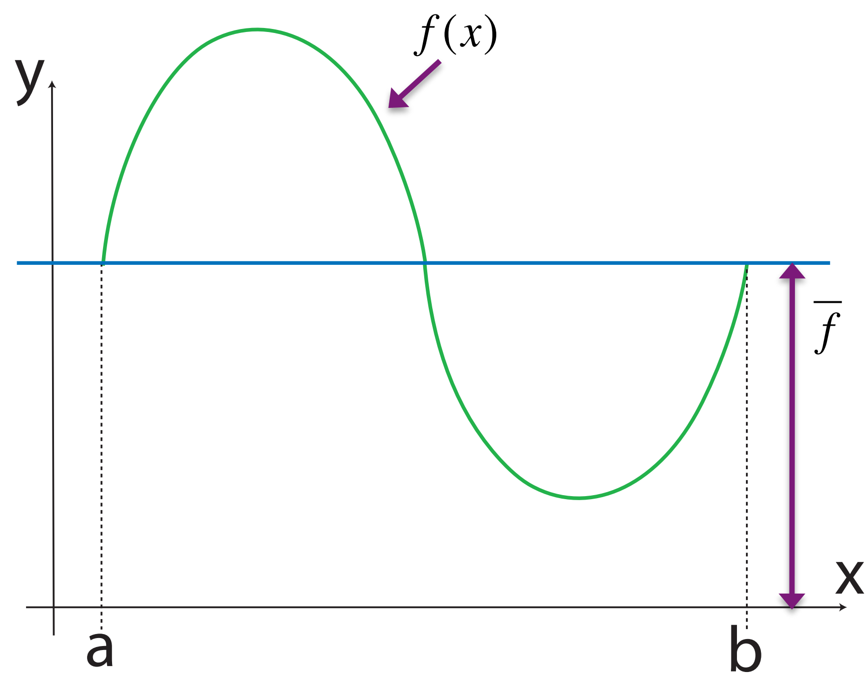
(a) $y = x^2 - 2x, \quad y = x + 4$

(b) $y = \sqrt{x}, \quad y = \dfrac{1}{x}, \quad x = \dfrac{1}{2}, \quad x = 2$

The Definite Integral - Average Value

The average value of a function f on the interval from a to b is

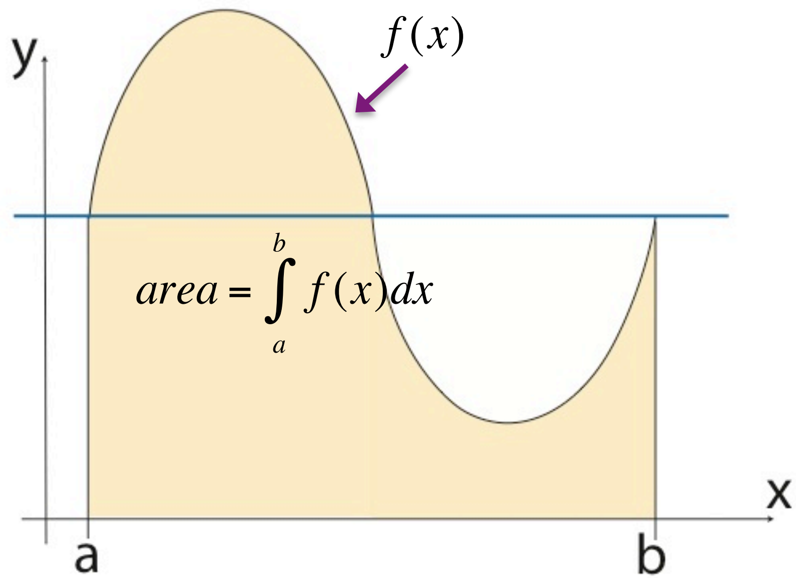
$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

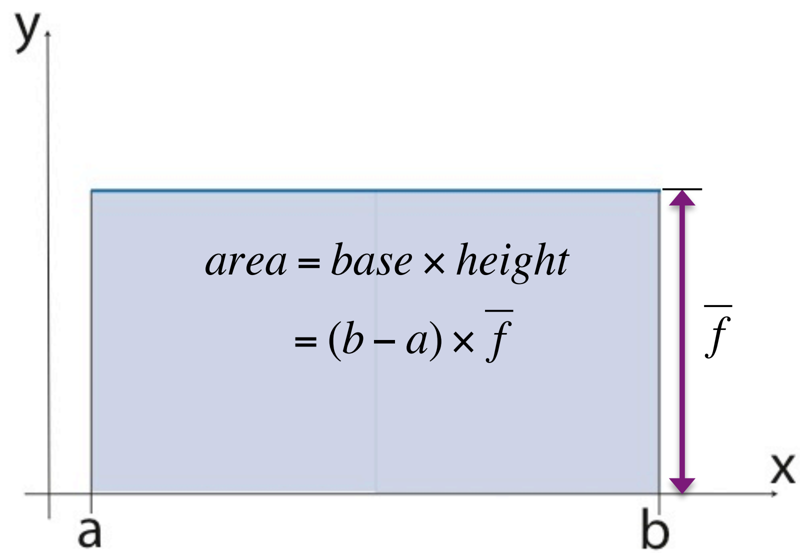


For a positive function,

$$\text{average height} = \frac{\text{area}}{\text{width}}$$

The Definite Integral - Average Value





$$\int_a^b f(x)\,dx = (b-a)\,\overline{f}$$

Application

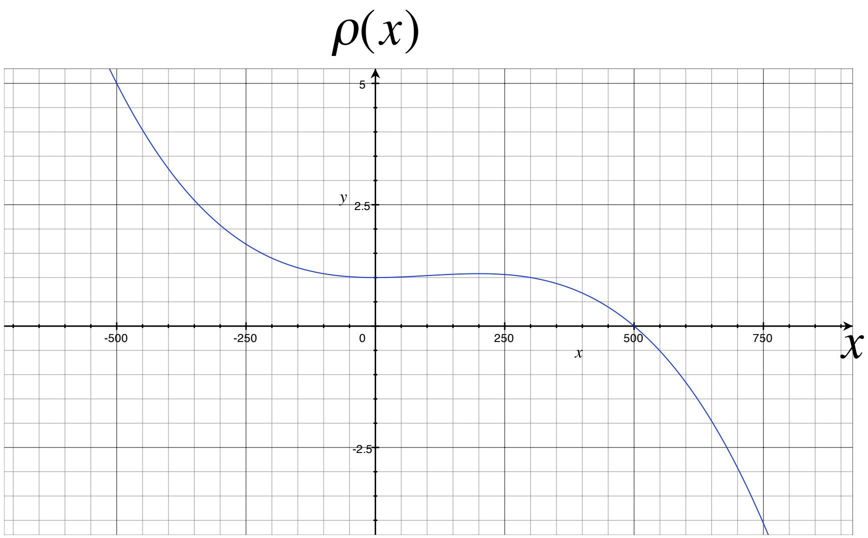
Example:

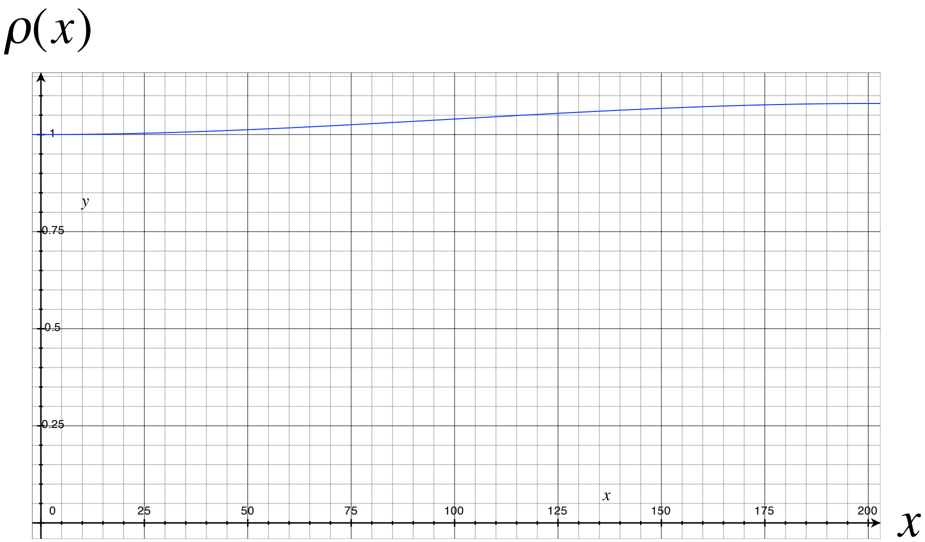
Several very skinny 2.0-m-long snakes are collected in the Amazon. Each snake has a density of

$$\rho(x) = 1 + 2 \times 10^{-8} x^2 (300 - x)$$

where ρ is measured in grams per centimeter and x is measured in centimeters from the tip of the tail.

Application





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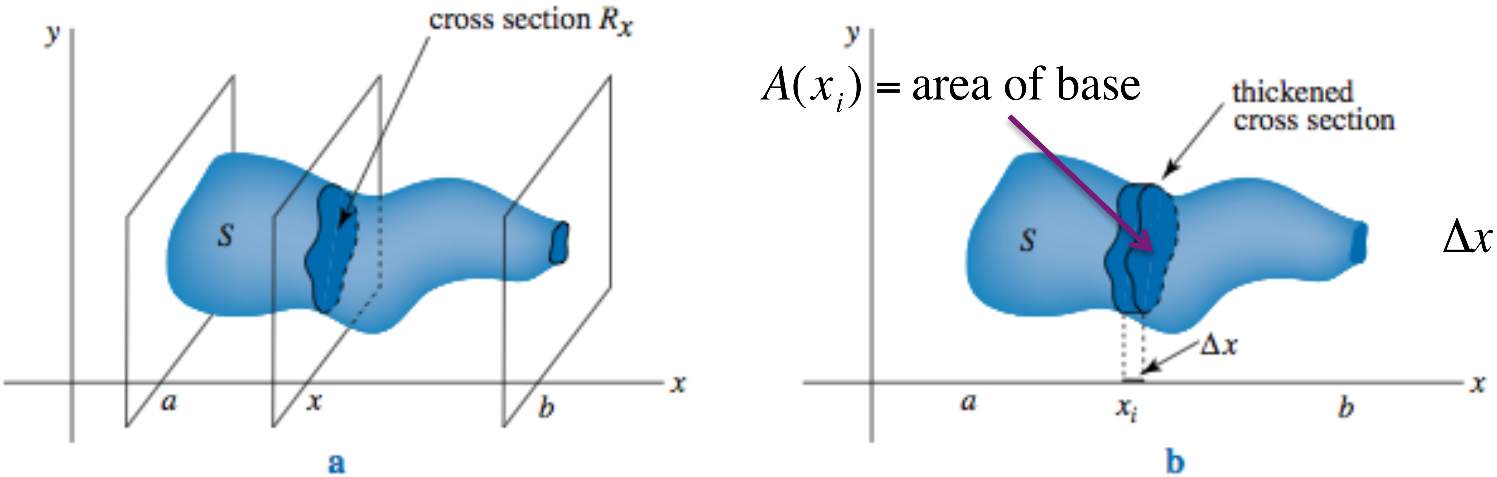
Application

(a) Find the total mass of each snake.

(b) Find the average density of each snake.

Approximating Volumes



$$\Delta x = \frac{b-a}{n}$$

$$V_n = A(x_1)\Delta x + A(x_2)\Delta x + \cdots + A(x_n)\Delta x$$

$$= \sum_{i=1}^{n} A(x_i)\Delta x$$

Riemann Sum

So, the volume V of the solid $S \approx V_n$.

Integrals and Volumes

Definition:
Denote by *A(x)* the area of the cross-section of *S* by the plane perpendicular to the *x*-axis that passes through *x*. Assume that *A(x)* is continuous on [*a,b*].

Then the **volume** *V* of *S* is given by

$$V = \lim_{n \to \infty} V_n = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i)\Delta x = \int_a^b A(x)\,dx$$

provided that the limit exists.

Volumes of Solids of Revolution

Examples:

Find the volume of the solid obtained by rotating the region *R* enclosed (bounded) by the given curves about the given axis.

(a) $y = \dfrac{1}{x}$, $y = 0$, $x = 1$, and $x = 2$ about the x-axis

(b) $y = 8 - x$, $y = 3$, $x = 2$, and $x = 5$ about the y-axis