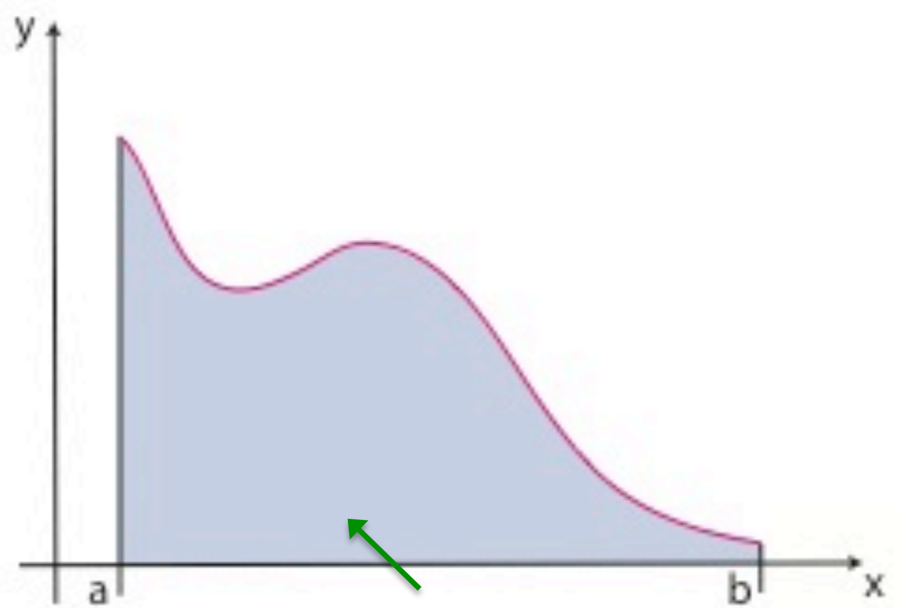
Definite (Proper) Integrals

Assumptions:

f is **continuous** on a **finite** interval [a,b].

$$\int_a^b f(x)\,dx$$ = real number

proper integral



Improper Integrals

Why are the following definite integrals "improper"?

$$\int_{1}^{\infty} \frac{1}{x^2}\,dx$$

$$\int_{-\infty}^{4} e^{-5x}\,dx$$

$$\int_{0}^{4} \frac{1}{x}\,dx$$

$$\int_{1}^{4} \frac{1}{(x-2)^2}\,dx$$

Improper Integrals
Type I: Infinite Limits of Integration

<u>*Definition*</u>:

Assume that the definite integral $\int_a^T f(x)\,dx$ exists (i.e., is equal to a real number) for every $T \geq a$. Then we define the improper integral of *f(x)* on (a, ∞) by

$$\int_a^\infty f(x)\,dx = \lim_{T\to\infty}\left(\int_a^T f(x)\,dx\right)$$

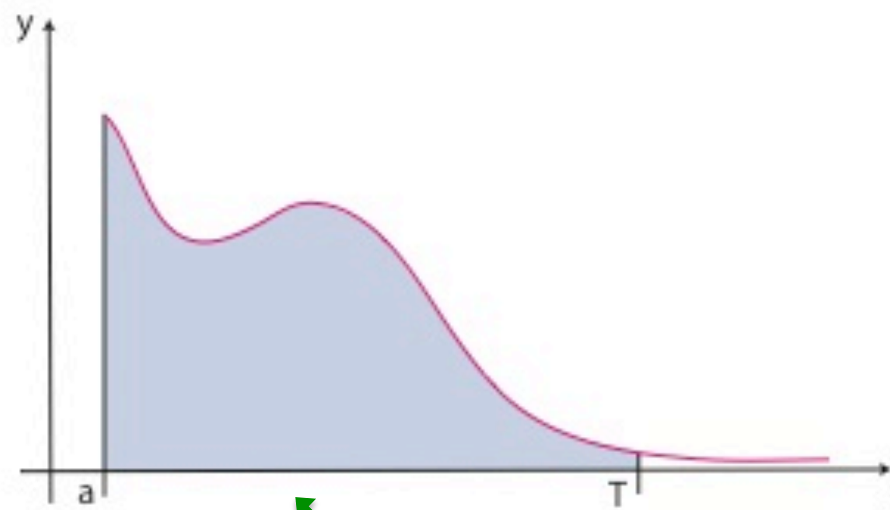
provided that the limit on the right side exists.

Improper Integrals
Type I: Infinite Limits of Integration

<u>Illustration:</u>

$$\int_a^{\infty} f(x)\,dx = \lim_{T\to\infty}\left(\int_a^{T} f(x)\,dx\right)$$

proper integral



finite region

Improper Integrals
Type I: Infinite Limits of Integration

Examples:

Evaluate the following improper integrals.

(a) $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$

(b) $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$

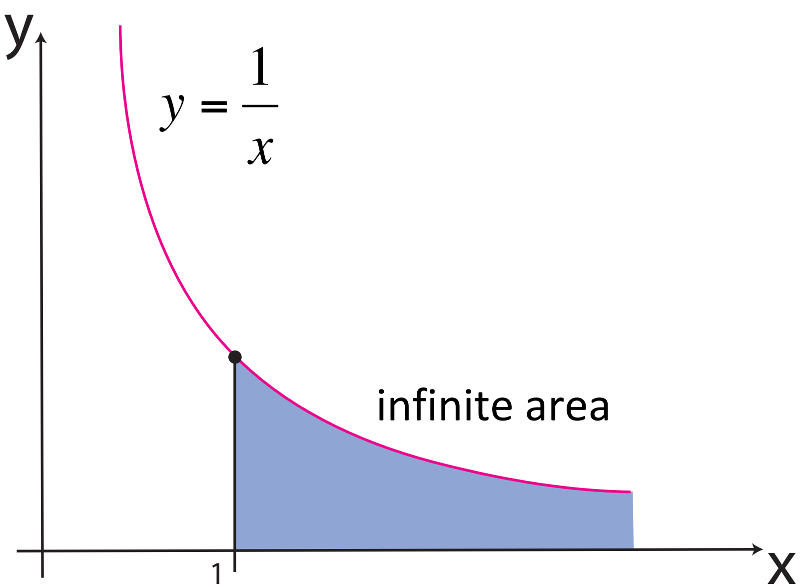
Improper Integrals
Type I: Infinite Limits of Integration

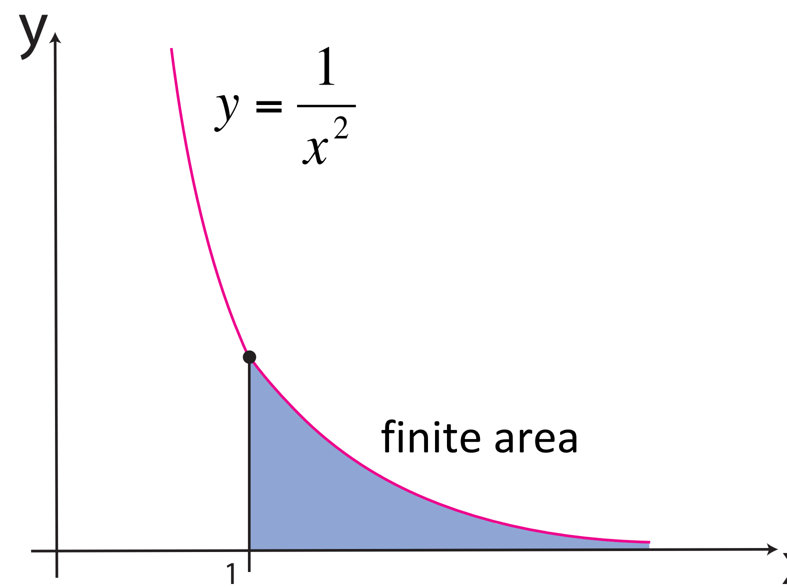
When the limit exists, we say that the integral *converges*.

When the limit does not exist, we say that the integral *diverges*.

Rule: $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p>1$ and divergent if $p\le 1$

Illustration





$\int_1^\infty \frac{1}{x} dx$ diverges

$\int_1^\infty \frac{1}{x^2} dx$ converges

Application

Example:
p. 584, #35.
The concentration of a toxin in a cell is increasing at a rate of $50e^{-2t}\ \mu mol/L/s,$ starting from a concentration of $10\,\mu mol/L.$ If the cell is poisoned when the concentration exceeds $30\,\mu mol/L,$ could this cell survive?

Improper Integrals
Type II: Infinite Integrands

Definition:

Assume that *f(x)* is continuous on (a,b] but not continuous at x=a. Then we define

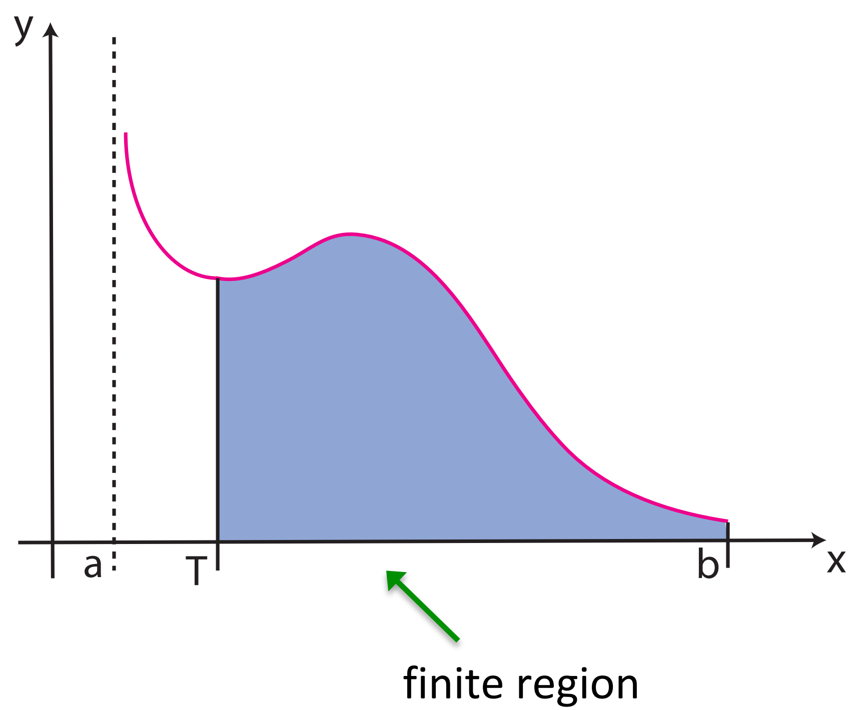
$$\int_a^b f(x)\,dx = \lim_{T \to a^+} \int_T^b f(x)\,dx$$

provided that the limit on the right side exists.

Improper Integrals
Type II: Infinite Integrands

<u>Illustration:</u>

$$\int_a^b f(x)\,dx = \lim_{T \to a^+} \left(\underbrace{\int_T^b f(x)\,dx}_{\text{proper integral}} \right)$$



Improper Integrals
Type II: Infinite Integrands

Examples:

Evaluate the following improper integrals.

(a) $\displaystyle\int_0^2 \frac{1}{\sqrt[3]{x}}\,dx$

(b) $\displaystyle\int_0^{10} \frac{1}{x^2}\,dx$

(c) $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

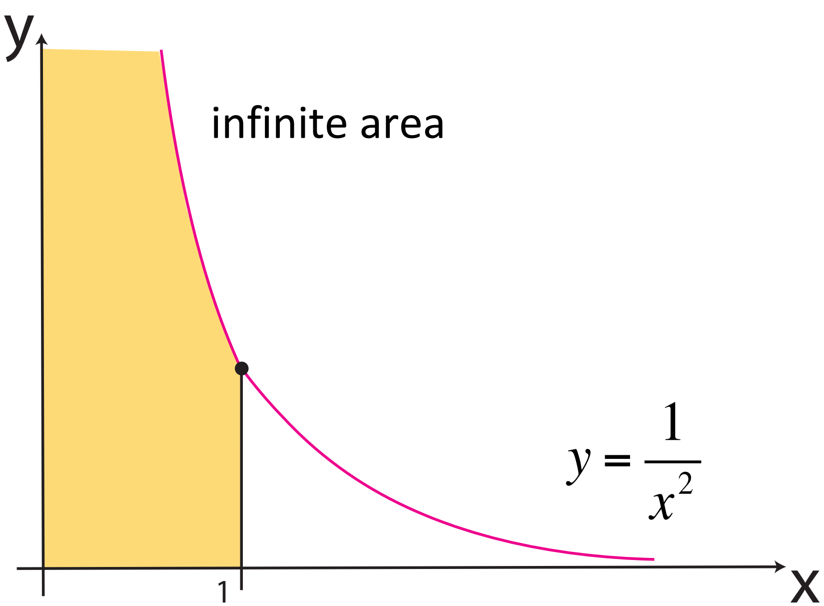
Improper Integrals
Type II: Infinite Integrands

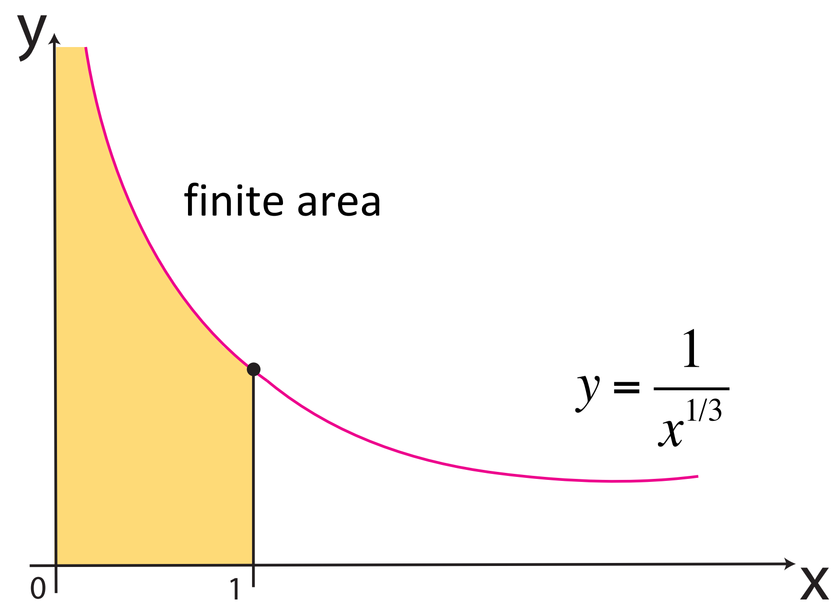
When the limit exists, we say that the integral *converges*.

When the limit does not exist, we say that the integral *diverges*.

Rule: $\int_0^1 \frac{1}{x^p}\,dx$ is convergent if $0 < p < 1$ and divergent if $p \geq 1$

Illustration





$\int_0^1 \frac{1}{x^2}\,dx$ diverges

$\int_0^1 \frac{1}{x^{1/3}}\,dx$ converges