Analysis of Autonomous Differential Equations

Section 8.1

Modelling

- Start with a simple model (differential equation) to roughly explain how a system changes then modify so it fits real-life observable data as close as possible.
- If you then observe an initial condition, you can use this rule (DE) to generate a solution and use it to predict future values.

Basic Exponential Model

Model:

$$\frac{dP}{dt} = k \cdot P(t)$$

P(t) = the number of individuals at time *t*
k = proportionality constant

Solution:

$$P(t) = P_0 e^{kt}$$

Basic Exponential Model

Example:

Suppose we know that the growth rate of a population is half of its current population. Then we have the model

$$\frac{dP}{dt} = 0.5P$$

Analyze the dynamics of this population.

Basic Exponential Model

Summary:

This model describes a population that grows at a rate proportional to its size. It assumes ideal conditions, i.e. unlimited resources, no predators, no disease, etc.

Logistic Model

Model:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

k = positive constant

L = carrying capacity

carrying capacity:
the maximum population that the environment is capable of sustaining in the long run

Logistic Model

Example:

A population grows according to the logistic model

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

Analyze the dynamics of this population.

Logistic Model

Notes:

1. The point at which there is a change in the pattern of increase is called an inflection point.

2. The population size at the point of inflection is one-half of the horizontal asymptote, i.e., one-half of the maximum population.

Logistic Model

Summary:

This model describes a population that grows exponentially for small values of P but as P increases, the growth rate slows down and the population approaches the carrying capacity.

If the population starts above its carrying capacity, it will decrease towards the carrying capacity.

Modified Logistic Differential Equation (the Allee Effect)

Model:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)\left(1 - \frac{m}{P}\right)$$

where

k , *m*, and *L* are positive constants and $m < L$

L = carrying capacity

m = existential threshold

Modified Logistic Differential Equation (the Allee Effect)

Example:

A population grows according to the modified logistic model

$$\frac{dP}{dt}=0.09P\left(1-\frac{P}{2000}\right)\left(1-\frac{120}{P}\right)$$

Analyze the dynamics of this population.

Modified Logistic Differential Equation (the Allee Effect)

Summary:

This model is similar to the logistic model but includes the idea of an existential threshold – the minimum number of individuals needed to sustain a population. If the population falls below this number, it will die out (decrease to 0).

Selection Model

Consider two variations of a certain population that grow at a rate proportional to their size.

$$\frac{da}{dt} = \mu a \qquad \frac{db}{dt} = \lambda b$$

a(t) = population size of type *a* at time *t*;

μ = per capita production rate of type *a*;

b(t) = population size of type *b* at time *t*;

λ = per capita production rate of type *b*.

Selection Model

It is often difficult to count the exact number of individuals for some populations, so instead we measure the *fraction* or *proportion* of each present in the total population.

$$p = fraction\ of\ type\ a = \frac{a}{a+b}$$

← # of individuals of type a

← total population size

$$1 - p = fraction\ of\ type\ b = \frac{b}{a+b}$$

Selection Model

The rate of change of the fraction of type a can be expressed as a logistic (autonomous) equation:

$$\frac{dp}{dt} = (\mu - \lambda) p (1 - p)$$

a measure of
the strength
of selection

<u>Calculations</u>:

Selection Model

Solution:

$$p(t) = \frac{p_0 e^{\mu t}}{p_0 e^{\mu t} + (1 - p_0) e^{\lambda t}}$$

where $p_0 = \frac{a_0}{a_0 + b_0}$

Calculations:

Selection Model

Example:
Suppose we find two strains of bacteria, type *a* and type *b*, where the per capita production for *a* is 0.5 and for *b* is 0.3.

(a) Write differential equations for the growth rate of each strain.

Selection Model

(b) Write an autonomous DE for *p*, the fraction of type a bacteria present in the sample.

(c) Given that initially 10% of the population is type *a*, write the solution for *p* and use it to find the fraction of type a bacteria present after 2 hours.

Selection Model

(d) Use Euler's Method with a step size of 1 to approximate the fraction of type *a* after 2 hours. (Compare to answer in (c))

Selection Model

(e) What happens as $t \to \infty$?

(f) Graph the solution.

Selection Model

Summary:

This model describes two variations of some population competing for the same resources. The rate of change of the fraction of type *a* is modeled by a logistic equation.

If the per capita production rate of type *a* is greater than that of type *b*, then type *a* will take over (i.e. the fraction of type a present will approach 1) and vice versa.