Equilibria and Phase-Line Diagrams for Autonomous DEs

Section 8.2

Equilibria

Definition:

A value, m^*, of the state variable is called an **equilibrium** of the autonomous differential equation

$$\frac{dm}{dt} = f(m)$$

if $f(m^*) = 0.$

Equilibria

Example:
Find the equilibria of the following autonomous DEs.

(a) $\dfrac{dx}{dt} = 1 - e^{x}$

(b) Selection Model

$$\frac{dp}{dt} = (\mu - \lambda)p(1 - p)$$

Phase-Line Diagrams

- A **phase-line diagram** is a graphical display of the qualitative behaviour of solutions to autonomous DEs.

- A phase-line diagram summarizes where the state variable is increasing, where it is decreasing, and where it is unchanged.

Phase-Line Diagrams

Construction:

The horizontal line represents the state variable.

For values of the state variable where the DE=0, large dots are drawn to indicate equilibrium solutions.

For values of the state variable where the DE is positive, right-pointing arrows are drawn to indicate that the solution is increasing.

For values of the state variable where the DE is negative, left-pointing arrows are drawn to indicate that the solution is decreasing.

The size of the arrows corresponds to the magnitude of the rate of change.

Phase-Line Diagrams

Example:

Newton's Law of Cooling

$$\frac{dT}{dt} = \alpha(A - T)$$

Phase-Line Diagrams

Example:
For the autonomous DE:

$$\frac{dx}{dt} = 1 - e^{x}$$

(i) Graph *dx/dt* as a function of *x*.
(ii) Create a phase-line diagram.
(iii) Draw some solution curves.

Phase-Line Diagrams

Example:
Draw a phase-line diagram for the modified logistic model

$$\frac{dP}{dt} = 0.09P\left(1-\frac{P}{2000}\right)\left(1-\frac{120}{P}\right)$$