Systems of Differential Equations

Section 8.5

Predator-Prey Model

Previously, we studied a variety of models for the growth of a single species that lives alone in an environment.

Now, we will consider two species interacting in the same habitat.

One species, called the **prey**, has an ample food supply and the second species, called the **predator**, feeds on the prey.

Predator-Prey Model

For example, consider a population of rabbits (prey) and wolves (predators) in an isolated forest.

Let

$$R(t) = \# \text{ rabbits at time } t$$

$$W(t) = \# \text{ wolves at time } t$$

Two dependent variables, both functions of time

Predator-Prey Model

<u>Assumption 1</u>:

Without predators, prey will grow exponentially.

$$\frac{dR}{dt} = kR, \quad k > 0$$

Without prey, predators will die out exponentially.

$$\frac{dW}{dt} = -rW, \quad r > 0$$

Predator-Prey Model

Assumption 2:

There will be more encounters between the two species if the population of either increases:

$$\#\ \text{of encounters} \propto RW$$

Predator-Prey Model

<u>*Assumption 3*</u>:

Encounters are bad for prey:

$$\frac{dR}{dt} = kR - aRW, \quad a > 0$$

Encounters are good for predators:

$$\frac{dW}{dt} = -rW + bRW, \quad b > 0$$

Predator-Prey Model

Predator-Prey Equations:

$$\frac{dR}{dt} = kR - aRW$$

$$\frac{dW}{dt} = -rW + bRW$$

W (for wolves) represents the predator

R (for rabbits) represents the prey

k, r, a, and b are positive constants

Predator-Prey Model

Example:
Suppose that the populations of rabbits and wolves are described by the predator-prey equations

$$\frac{dR}{dt} = 0.08R - 0.001RW$$

$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

where time *t* is measured in months.

Predator-Prey Model

Example:

(a) Graph the per capita growth rates for each species.

(b) Suppose that initially there are 1000 rabbits and 40 wolves. What will happen to these populations after 2 months? (use Euler's Method)

Euler's Method for a Pair of Linked Autonomous DEs

Algorithm:

$$t_{n+1} = t_n + h$$
$$x_{n+1} = x_n + f(x_n, y_n)h$$
$$y_{n+1} = y_n + g(x_n, y_n)h$$

Algorithm In Words:

next time step = previous time step + step size

next approximation = previous approximation + rate of change of the function at previous value**s** x step size

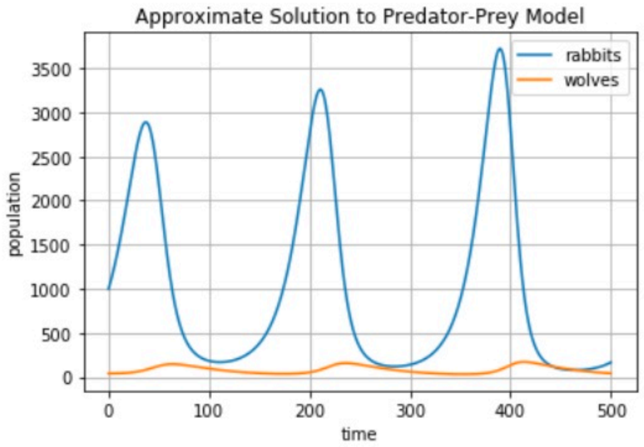
Predator-Prey Model

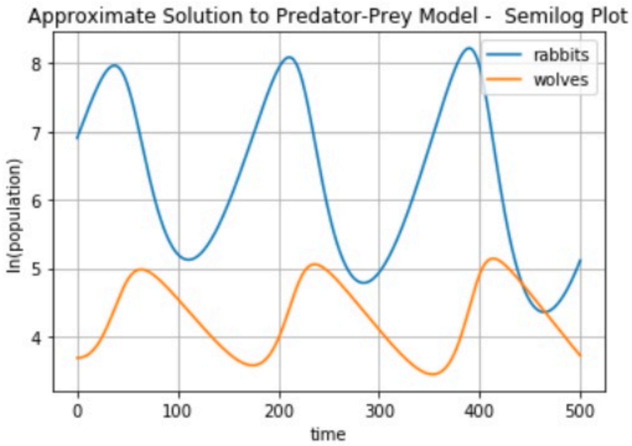
```
# predator-prey model

n = 2 # number of iterations/steps
h = 1 # step size
k = 0.08
a = 0.001
r = 0.02
b = 0.00002

t = [0] # array for t with initial condition entered
R = [1000] # array for R with initial condition entered
W = [40] # array for W with initial condition entered

for i in range(1, n+1):
    ti = t[i-1] + h
    Ri = R[i-1] + (k*R[i-1]-a*R[i-1]*W[i-1])*h
    Wi = W[i-1] + (-r*W[i-1]+b*R[i-1]*W[i-1])*h
    t.append(ti)
    R.append(Ri)
    W.append(Wi)

print("time:", t)
print("rabbit popultion:", R)
print("wolf popultion:", W)
```

```
time: [0, 1, 2]
rabbit popultion: [1000, 1040.0, 1081.6]
wolf popultion: [40, 40.0, 40.032]
```

Predator-Prey Model

```
# predator-prey model

n = 500 # number of iterations/steps
h = 1 # step size
k = 0.08
a = 0.001
r = 0.02
b = 0.00002

t = [0] # array for t with initial condition entered
R = [1000] # array for R with initial condition entered
W = [40] # array for W with initial condition entered

for i in range(1, n+1):
    ti = t[i-1] + h
    Ri = R[i-1] + (k*R[i-1]-a*R[i-1]*W[i-1])*h
    Wi = W[i-1] + (-r*W[i-1]+b*R[i-1]*W[i-1])*h
    t.append(ti)
    R.append(Ri)
    W.append(Wi)

# plot solution
plt.plot(t,R, label = 'rabbits')
plt.plot(t,W, label = 'wolves')
plt.title("Approximate Solution to Predator-Prey Model")
plt.xlabel("time")
plt.ylabel("population")
plt.legend()
plt.grid()
plt.show()

# semilog plot
plt.plot(t,np.log(R), label = 'rabbits')
plt.plot(t,np.log(W), label = 'wolves')
plt.title("Approximate Solution to Predator-Prey Model -  Semilog Plot")
plt.xlabel("time")
plt.ylabel("ln(population)")
plt.legend()
plt.grid()
plt.show()
```





Note: This code is posted on our course webpage for you to explore!

Systems of Differential Equations

The predator-prey model is an example of a system of **coupled** (or **linked**) autonomous differential equations.

<u>Coupled Autonomous Differential Equations</u>:
A pair of differential equations in which the rate of change of each state variable depends on its own value and on the value of the other state variable.

$$\frac{dx}{dt} = f(x,y) \quad \text{and} \quad \frac{dy}{dt} = g(x,y)$$

Competitive Selection Model

Recall: **Selection Model**

When two variations of a certain population grow at a rate proportional to their size, we can write a pair of *uncoupled* autonomous DEs:

$$\frac{da}{dt} = \mu a \qquad \frac{db}{dt} = \lambda b$$

a(t)=population size of type *a* at time *t*;
μ =per capita production rate of type *a*;

b(t)=population size of type *b* at time *t*;
λ =per capita production rate of type *b*.

Competitive Selection Model

Consider the case in which these two types interact and compete for the same resources.

As the size of the total population increases, so does competition for resources, which has a negative effect on the growth rate for each type.

Competitive Selection Model

Suppose that the per capita growth rate of each type decreases *linearly* as a function of the total population, *a*+*b*:

$$\text{per capita growth rate of type } a = \mu\left(1 - \frac{a+b}{K_a}\right)$$

$$\text{per capita growth rate of type } b = \lambda\left(1 - \frac{a+b}{K_b}\right)$$

$$K_a = \text{carrying capacity of type } a$$

$$K_b = \text{carrying capacity of type } b$$

Competitive Selection Model

The coupled autonomous DEs for a competitive selection model are given by

$$\frac{da}{dt} = \mu\left(1 - \frac{a+b}{K_a}\right)a \quad \text{and} \quad \frac{db}{dt} = \lambda\left(1 - \frac{a+b}{K_b}\right)b$$

where $K_a = \text{carrying capacity of type } a$

$K_b = \text{carrying capacity of type } b$

Competitive Selection Model

Example:

Suppose $K_a = 100$ and $K_b = 200$.

(a) Graph the per capita growth rates for *a* and *b* as functions of the total population, $a+b$.

(b) If $a_0 = 50$ and $b_0 = 100$, what would happen to the size of each population in the immediate future?

Newton's Law of Cooling

<u>Recall</u>:

Newton's law of cooling expresses the rate of change of the temperature, *T*, of an object as a function of the ambient temperature, *A*, by the equation

$$\frac{dT}{dt} = \alpha(A - T)$$

where α depends on the the size, shape, and material of the object.

Newton’s Law of Cooling

If the object is large relative to its environment, it will also have an effect on the ambient temperature.

Newton’s Law of Cooling can then be applied to describe the rate of change of the ambient temperature by the equation

$$\frac{dA}{dt} = \alpha_2 (T - A)$$

where α_2 depends on the the size, shape, and heat properties of the environment the object is in.

Newton's Law of Cooling

The rates of change of the temperature of the object and its environment are given by the following system of **coupled autonomous DEs:**

$$\frac{dT}{dt} = \alpha(A - T) \quad \text{and} \quad \frac{dA}{dt} = \alpha_2(T - A)$$

In general, α_2 will be smaller as the environment becomes larger.

A Model for a Disease

<u>Recall</u>: **<u>Basic Model for a Disease</u>**

Suppose a disease is circulating in a population. Individuals recover from this disease unharmed but are susceptible to reinfection.

Let I denote the fraction of infected individuals in a population. Then, the rate at which the fraction of infected individuals is changing is given by

$$\frac{dI}{dt} = \alpha I(1 - I) - \mu I$$

where α and μ are positive constants.

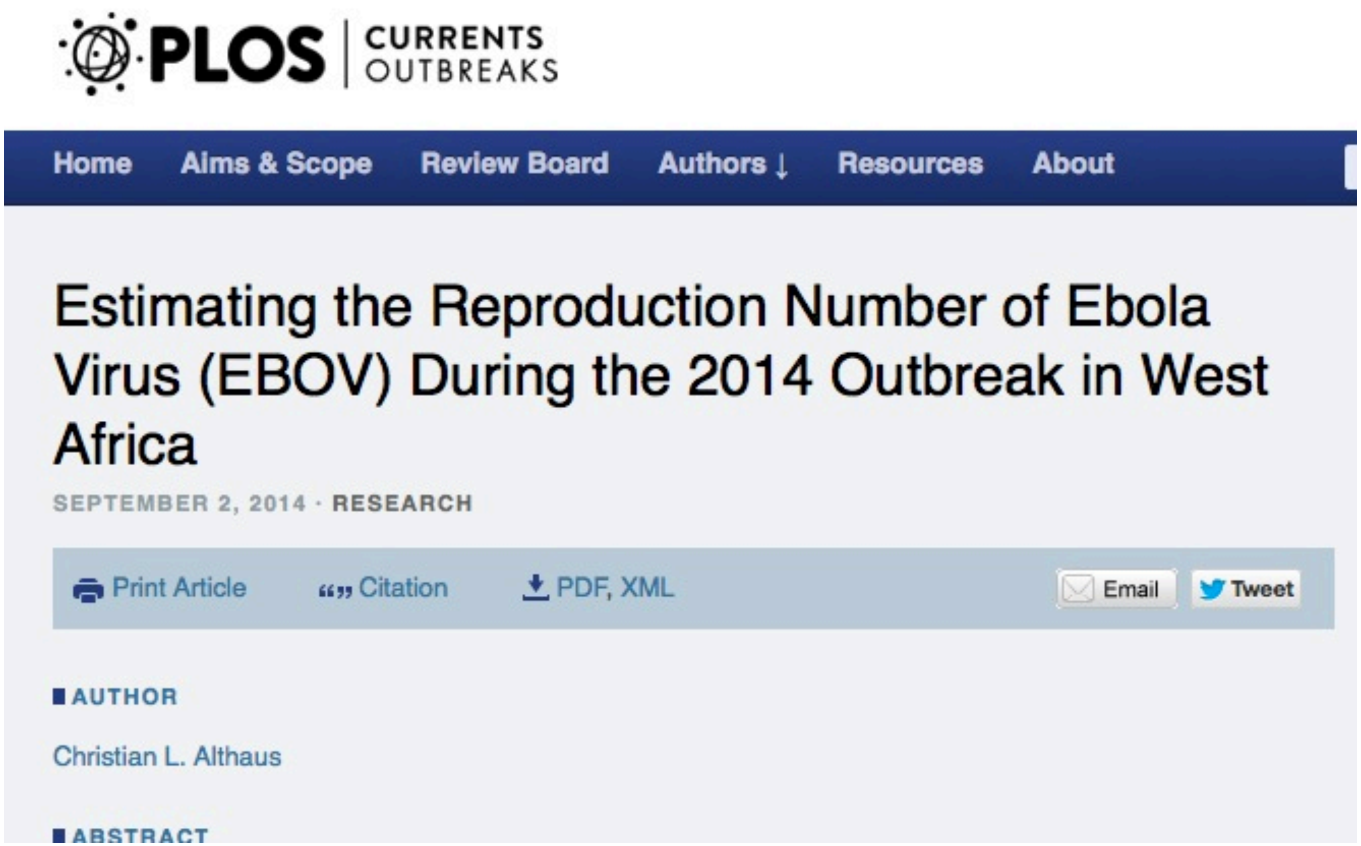
A Model for a Disease

Exercise:

Keeping the assumptions of the basic model for a disease, write a pair autonomous differential equations to describe how both the proportion of infected individuals, I, and the proportion of susceptible individuals, S, change over time.

A Model for a Disease

Recall: This journal article on the Ebola virus (EBOV) outbreak in West Africa was studied in Math 1LS3



A Model for a Disease

■ **METHODS**

The transmission of EBOV follows SEIR (susceptible-exposed-infectious-recovered) dynamics and can be described by the following set of ordinary differential equations (ODEs):[2]

$$\frac{dS}{dt} = -\beta(t)SI/N,$$
$$\frac{dE}{dt} = \beta(t)SI/N - \sigma E,$$
$$\frac{dI}{dt} = \sigma E - \gamma I,$$
$$\frac{dR}{dt} = (1-f)\gamma I.$$

After transmission of the virus, susceptible individuals S enter the exposed class E before they become infectious individuals I that either recover and survive (R) or die. $1/\sigma$ and $1/\gamma$ are the average durations of incubation and infectiousness. The case fatality rate is given by f. The transmission rate in absence of control interventions is constant, i.e., $\beta(t) = \beta$. After control measures are introduced at time $\tau \le t$, the transmission rate was assumed to decay exponentially at rate k:[3]

$$\beta(t) = \beta e^{-k(t-\tau)},$$

i.e., the time until the transmission rate is at 50% of its initial level is $t_{1/2} = \ln(2)/k$. Assuming the epidemic