The Phase Plane

Section 8.6

The Phase Plane

The **phase plane** for a system of two autonomous DEs is a coordinate plane with the axes representing the values of the two state variables.

A **phase-plane diagram** is a graphical display of the qualitative behaviour of solutions to a system of two autonomous DEs.

Nullclines

A **nullcline** is a set of points for which a state variable does not change (i.e., for which the rate of change of the state variable is zero).

For the system $\dfrac{dx}{dt} = f(x,y)$ and $\dfrac{dy}{dt} = g(x,y)$

the solutions of $\dfrac{dx}{dt} = f(x,y) = 0$ define the ***x*-nullcline** and

the solutions of $\dfrac{dy}{dt} = g(x,y) = 0$ define the ***y*-nullcline**.

Predator-Prey Model

Example:

Find and graph the *R*- and *W*-nullclines in the phase plane for the predator-prey model

$$\frac{dR}{dt} = 0.08R - 0.001RW$$

$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

Equilibria

An **equilibrium** of a two-dimensional system of autonomous DEs is a point where the rate of change of *both* state variables is zero.

Equilibria can be found by solving the system

$$\frac{dx}{dt} = f(x,y) = 0 \quad \text{and} \quad \frac{dy}{dt} = g(x,y) = 0$$

Predator-Prey Model

Example:

Identify the equilibria of the predator-prey model:

$$\frac{dR}{dt} = 0.08R - 0.001RW$$

$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

Finding the Nullclines and Equilibria of Coupled Autonomous DEs

Algorithm:

1. Decide which variable is represented by the horizontal axis and which one by the vertical axis in the phase plane.
2. Write the equations for the nullclines and solve them.
3. Graph each solution in the phase plane.
4. Identify the intersections of nullclines belonging to different variables, as these are the equilibria of the system.

Modified Competition Equations

Example: Graph the nullclines and find the equilibria for the modified competition equations:

$$\frac{da}{dt} = \mu\left(1 - \frac{a+b}{K_a}\right)a, \quad \frac{db}{dt} = \lambda\left(1 - \frac{b}{K_b}\right)b$$