Solutions in the Phase Plane

Section 8.7

Solutions

Our ultimate goal is to find *solutions*, that is, descriptions of how the state variables change over time.

Since systems of autonomous differential equations are generally impossible to solve exactly, we need to use alternative methods to determine the behaviour of solutions.

Solutions in the Phase Plane

One approach is to approximate values of solutions using a numerical method, such as Euler's method.

Example:

Approximate values of the solutions b(t) and p(t) for the predator-prey equations

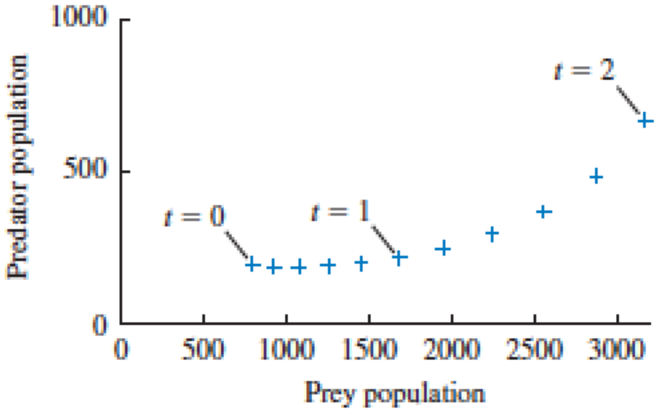
$$\frac{db}{dt} = (1 - 0.001p)b, \quad \frac{dp}{dt} = (-1 + 0.001b)p$$

Solutions in the Phase Plane

Table

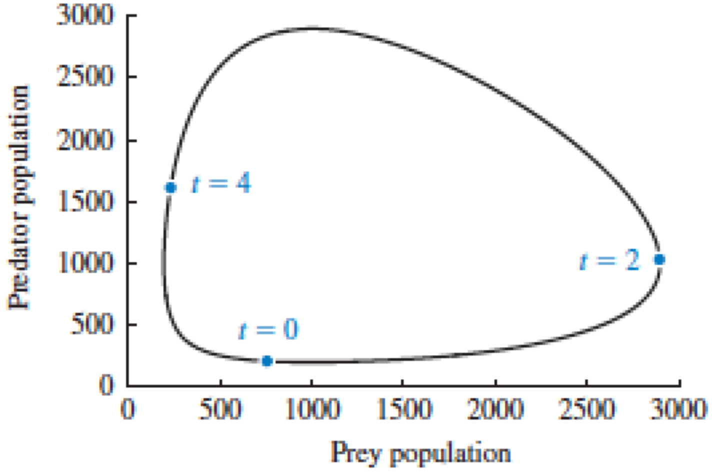
t	Approximation for b	Approximation for p
0	800	200
0.2	928	192
0.4	1078	189.2
0.6	1252.8	192.2
0.8	1455.2	201.9
1	1687.4	220.3
1.2	1950.6	250.6
1.4	2242.9	298.2
1.6	2557.8	372.3
1.8	2878.8	488.3
2	3173.4	671.8

Results from Euler's method plotted in the phase plane



Solutions in the Phase Plane

The graph of a solution in the phase plane is called a **phase-plane trajectory**.



Solutions in the Phase Plane

To describe solutions qualitatively, and avoid the calculations required for Euler's method, we can add *direction arrows* to our phase-plane diagram.

Using these direction arrows, we can sketch a phase-plane trajectory, starting from given initial conditions.

Direction Arrows

In a phase-plane diagram, nullclines divide the plane into regions. We can determine the behaviour of the state variables in each region and draw a *direction arrow* to simultaneous represent these behaviours.

Note: Direction arrows on nullclines are either vertical or horizontal.

Direction Arrows

Drawing Direction Arrows:

Method 1: Pick a pair of values, (b,p), in the region and substitute into the DE. Check whether db/dt and dp/dt are positive or negative.

Method 2: Manipulate the inequalities algebraically.

Method 3: Reason about the equations.

Solutions in the Phase Plane

Example:

(a) Add direction arrows to the phase-plane diagram for the predator-prey equations

$$R' = 0.08R - 0.001RW$$
$$W' = -0.02W + 0.00002RW$$

(b) Use the direction arrows to sketch a phase-plane trajectory starting from (R,W) = (1000, 40).

Solutions in the Phase Plane

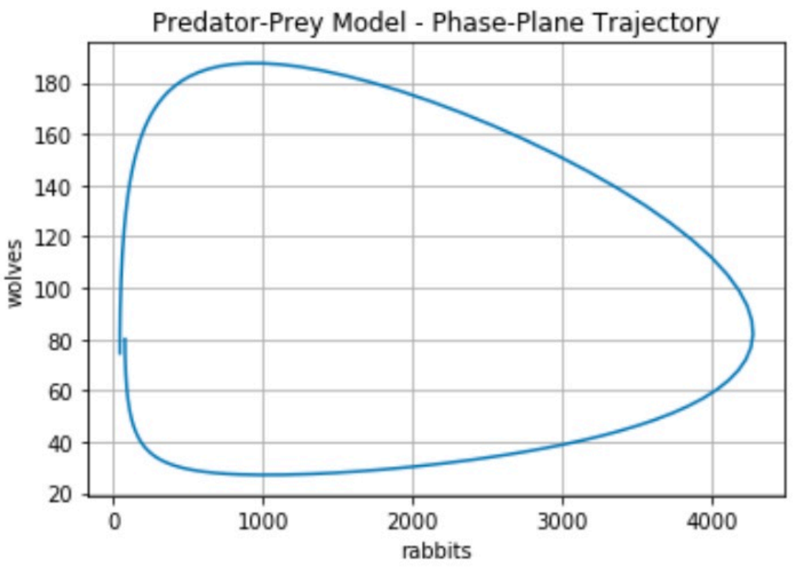
```
# predator-prey model

n = 190 # number of iterations/steps
h = 1 # step size
k = 0.08
a = 0.001
r = 0.02
b = 0.00002

t = [0] # array for t with initial condition entered
R = [80] # array for R with initial condition entered
W = [80] # array for W with initial condition entered

for i in range(1, n+1):
    ti = t[i-1] + h
    Ri = R[i-1] + (k*R[i-1]-a*R[i-1]*W[i-1])*h
    Wi = W[i-1] + (-r*W[i-1]+b*R[i-1]*W[i-1])*h
    t.append(ti)
    R.append(Ri)
    W.append(Wi)

# plot phase-plane trajectory
plt.plot(R,W)
plt.title("Predator-Prey Model - Phase-Plane Trajectory")
plt.xlabel("rabbits")
plt.ylabel("wolves")
plt.grid()
plt.show()
```



Modified Competition Equations

Example: Draw a phase-plane diagram for the modified competition equations:

$$\frac{da}{dt} = \mu\left(1 - \frac{a+b}{200}\right)a, \quad \frac{db}{dt} = \lambda\left(1 - \frac{b}{100}\right)b$$

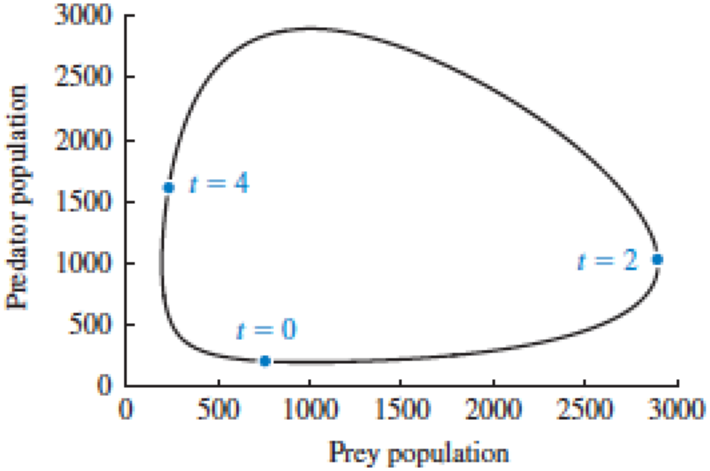
Add direction arrows to your diagram and use them to draw phase-plane trajectories starting in each region.

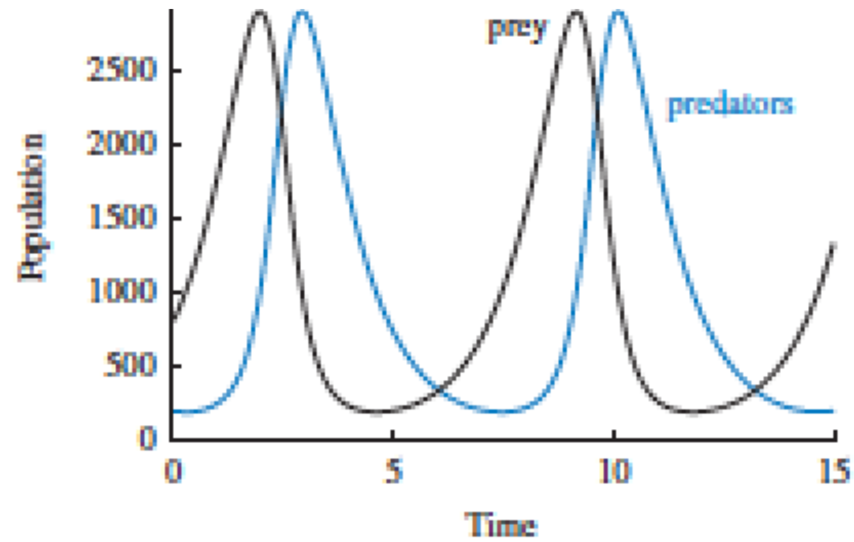
Phase-Plane Trajectory ➡ Graph of a Solution

Starting from the phase-plane trajectory, we can sketch the solutions by tracing along the graph at a constant speed. The horizontal location of the pencil gives the prey population, or the height of the graph $b(t)$. The vertical location of the pencil gives the predator population, or the height of the graph $p(t)$.

Phase-Plane Trajectory → Graph of a Solution

Example:





Graph of a Solution ➡ Phase-Plane Trajectory

Starting from the solution, we can plot the number of predators against the number of prey at several times (such as t=0, t=1, etc.) and connect the dots.

Graph of a Solution ➡ Phase-Plane Trajectory

Example: