The Binomial Distribution

Section 10

Bernoulli Experiment and Bernoulli Random Variable

A *Bernoulli experiment* is a random experiment with only two possible outcomes: success or no-success.

Definition:

A discrete random variable that takes on the value 1 (“success”) with probability p and the value 0 (“no-success”) with probability $1-p$ is called a *Bernoulli random variable*.

The Binomial Distribution

Let N count the number of successes in n repetitions of the same Bernoulli experiment, where outcomes are independent and p is the probability of success in a single experiment.

Then N is a binomially distributed random variable and we write

$$N \sim B(n,p)$$

The Binomial Distribution

Define the *binomial probability distribution* by

$$b(k, n; p) = P(N = k)$$

where $b(k, n; p)$ is the probability of exactly k successes in n repetitions of the same experiment, where p is the probability of success in a single experiment.

Exercise

Example: Elephant Population with Immigration

Consider the population of elephants p_t modelled by

$$p_{t+1} = p_t + I_t \quad \text{where } I_t = \begin{cases} 10 & \text{with a 90\% chance} \\ 0 & \text{with a 10\% chance} \end{cases}$$

where $t = 0, 1, 2, \ldots$ is measured in years.

Let N count the number of times immigration occurs over the next 3 years. Determine the probability mass function for N.

Exercise

Example: Elephant Population with Immigration

Using the same approach as in the previous exercise, determine the probability mass function for *N,* where *N* is the number of times the population increases by 10 elephants over the **next 4 years**.

The Binomial Distribution

The probability of *k* successes in *n* experiments is (number of ways of obtaining *k* successes in *n* experiments)*(probability of success)^{k}* (probability of no-success)$^{n\text{-}k}$

i.e.,

$$b(k, n; p) = C(n,k)\,p^{k}(1-p)^{n-k}$$

Counting 101

Suppose we have a selection of 10 books.

1. How many different orderings (permutations) can you read them in?

2. How many different orderings can you read 3 in?

3. Suppose you want to read three books but you don't care about the order in which you read them. How can you choose 3 books from 10?

Counting 101

Now, replace "10 books" by *"n* experiments" and "choose 3 books" by "choose *k* experiments in which there is a success".

The numbers of ways we can have *k* successes in *n* repetitions of the experiment is nCk. So, C(n,k)=nCk.

The Probability Distribution of the Binomial Variable

Theorem:

The probability distribution of the binomial variable *N* is given by

$$P(N = k) = b(k, n; p) = \binom{n}{k} p^k (1 - p)^{n-k}$$

where *N* counts the number of successes in *n* independent repetitions of the same Bernoulli experiment and *p* is the probability of success.

The Probability Distribution of the Binomial Variable

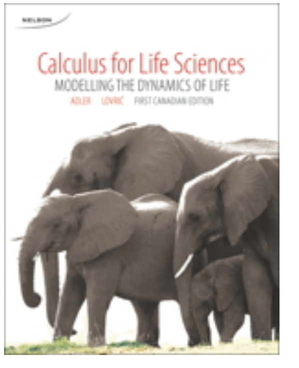
Example: Elephant Population with Immigration

Consider a population of elephants p_t modelled by

$$p_{t+1} = p_t + I_t \quad \text{where} \quad I_t = \begin{cases} 10 & \text{with a 90\% chance} \\ 0 & \text{with a 10\% chance} \end{cases}$$

Suppose that initially there are 80 elephants.

What is the probability that there will be more than 300 elephants after 25 years?



The Mean and Variance of the Binomial Distribution

Mean and Variance of the Binomial Random Variable *N*:

$$E(N) = np$$

$$Var(N) = np(1-p)$$

Exercise

Example: Elephant Population with Immigration

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$$p_{t+1} = p_t + I_t \quad \text{where} \quad I_t = \begin{cases} 10 & \text{with a } 90\% \text{ chance} \\ 0 & \text{with a } 10\% \text{ chance} \end{cases}$$

Suppose that initially there are 80 elephants.

What is the expected value of the population after 25 years? What is the standard deviation?