The Normal Distribution

Section 14

The Normal Distribution

The normal distribution is the most important continuous distribution as it can be used to model many phenomena in a variety of fields.

Many measurements for large sample sizes are said to be ‘normally distributed.’

For example, heights of trees, IQ scores, and duration of pregnancy are all normally distributed measurements.

The Normal Distribution

Definition:

A continuous random variable *X* has a normal distribution (or is distributed normally) with mean μ and variance σ^2, denoted by $X \sim N(\mu, \sigma^2)$, if its probability density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where $x \in (-\infty, \infty)$.

The Normal Distribution

The graph of the probability density function of the normal distribution (also known as the Gaussian distribution) is a bell-shaped curve.

Properties of the Normal Distribution Density Function

Theorem:

The probability density function *f(x)* of the normal distribution satisfies the following properties:

(a) f(x) is symmetric with respect to the vertical line $x = \mu$.

(b) *f(x)* is increasing for $x < \mu$ and decreasing for $x > \mu$. It has a local (also global) maximum value $1/\sigma\sqrt{2\pi}$ at $x = \mu$.

(c) The inflection points of *f(x)* are $x = \mu \pm \sigma$.

(d) $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 0$

Calculating Probabilities

If *X* is a normally distributed continuous random variable with mean μ and variance σ^2, then

$$P(a \le X \le b) = \int_a^b f(x)dx = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Calculating Probabilities

This integral cannot be evaluated without estimation techniques, such as using a Taylor polynomial to approximate $f(x)$.

To evaluate this integral, we reduce a general normal distribution to a special normal distribution, called the standard normal distribution, and then use tables of estimated values.

Standard Normal Distribution

Definition:

The standard normal distribution is the normal distribution with mean 0 and variance 1; in symbols, it is *N(0,1)*. Its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

for all $x \in (-\infty, \infty)$.

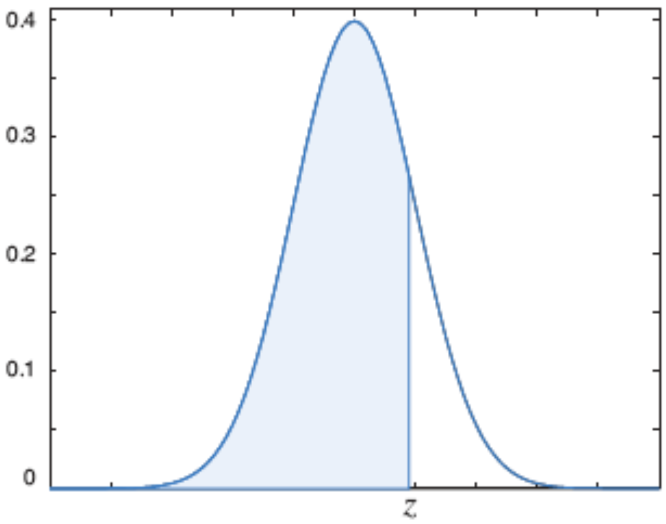
Standard Normal Distribution

We use the symbol *Z* to denote the continuous random variable that has the standard normal distribution; i.e., $Z \sim N(0,1)$.

Standard Normal Distribution

The cumulative distribution function of *Z* is given by

$$F(z) = \int_{-\infty}^{z} f(x)dx = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$



$$F(-z) = 1 - F(z)$$

z	$F(z)$	z	$F(z)$	z	$F(z)$	z	$F(z)$
0	0.500000	1	0.841345	2	0.977250	3	0.998650
0.05	0.519938	1.05	0.853141	2.05	0.979818	3.05	0.998856
0.1	0.539828	1.1	0.864334	2.1	0.982136	3.1	0.999032
0.15	0.559618	1.15	0.874928	2.15	0.984222	3.15	0.999184
0.2	0.579260	1.2	0.884930	2.2	0.986097	3.2	0.999313
0.25	0.598706	1.25	0.894350	2.25	0.987776	3.25	0.999423
0.3	0.617911	1.3	0.903200	2.3	0.989276	3.3	0.999517
0.35	0.636831	1.35	0.911492	2.35	0.990613	3.35	0.999596
0.4	0.655422	1.4	0.919243	2.4	0.991802	3.4	0.999663
0.45	0.673645	1.45	0.926471	2.45	0.992857	3.45	0.999720
0.5	0.691462	1.5	0.933193	2.5	0.993790	3.5	0.999767
0.55	0.708840	1.55	0.939429	2.55	0.994614	3.55	0.999807
0.6	0.725747	1.6	0.945201	2.6	0.995339	3.6	0.999840
0.65	0.742154	1.65	0.950529	2.65	0.995975	3.65	0.999869
0.7	0.758036	1.7	0.955435	2.7	0.996533	3.7	0.999892
0.75	0.773373	1.75	0.959941	2.75	0.997020	3.75	0.999912
0.8	0.788145	1.8	0.964070	2.8	0.997445	3.8	0.999928
0.85	0.802337	1.85	0.967843	2.85	0.997814	3.85	0.999941
0.9	0.815940	1.9	0.971283	2.9	0.998134	3.9	0.999952
0.95	0.828944	1.95	0.974412	2.95	0.998411	3.95	0.999961
						4	0.999968

Standard Normal Distribution

Example:

Determine the following:

(a) $F(0)$ (b) $F(0.18)$ (c) $F(-1)$

The Normal and the Standard Normal Distributions

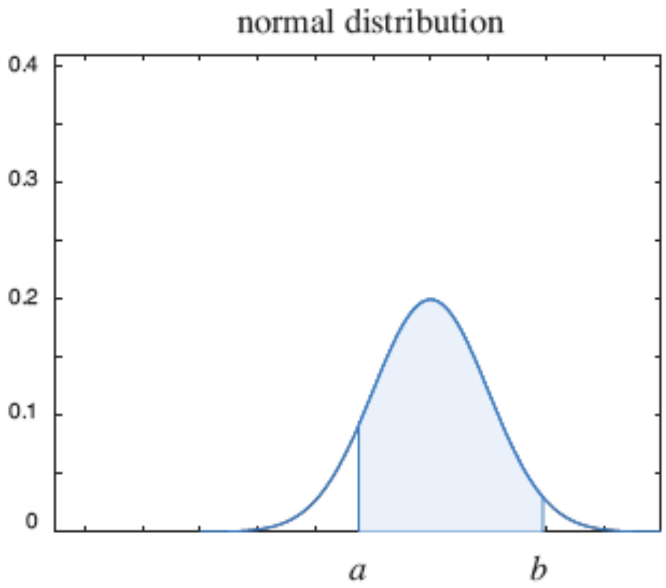
Theorem:

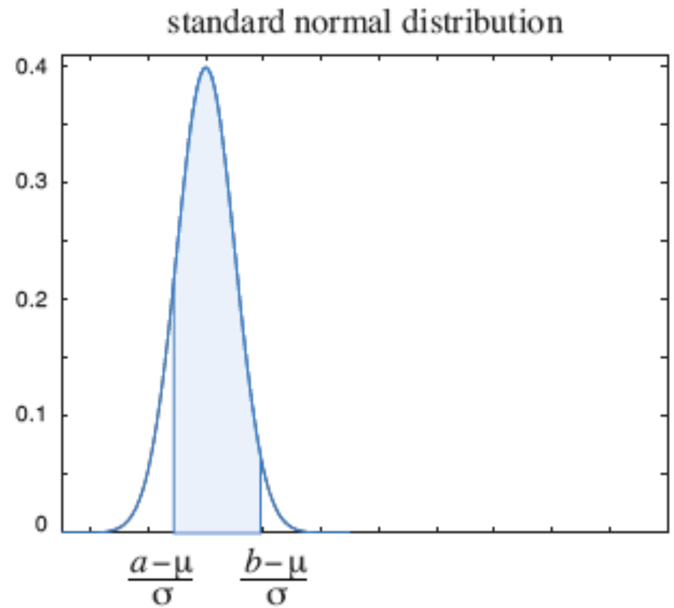
Assume that $X \sim N(\mu, \sigma^2)$. The random variable $Z = (X - \mu)/\sigma$ has the standard normal distribution, i.e., $Z \sim N(0,1)$.

So then $P(a \le X \le b) = P(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma})$.

The Normal and the Standard Normal Distributions

In words, the area under the normal distribution density function between *a* and *b* is equal to the area under the standard normal density function between $(a-\mu)/\sigma$ and $(b-\mu)/\sigma$.





The Normal and the Standard Normal Distributions

Example #10:

Let $X \sim N(-2,4)$; find $P(-3 \le X \le 1)$.

Example #30:

Let $X \sim N(2,144)$; find a value of x that satisfies $P(X > x) = 0.3$.

Application

Example:

Intelligence quotient (IQ) scores are distributed normally with mean 100 and standard deviation 15.

(a) What percentage of the population has an IQ score between 85 and 115?

(b) What percentage of the population has an IQ above 140?

(c) What IQ score do 90% of people fall under?

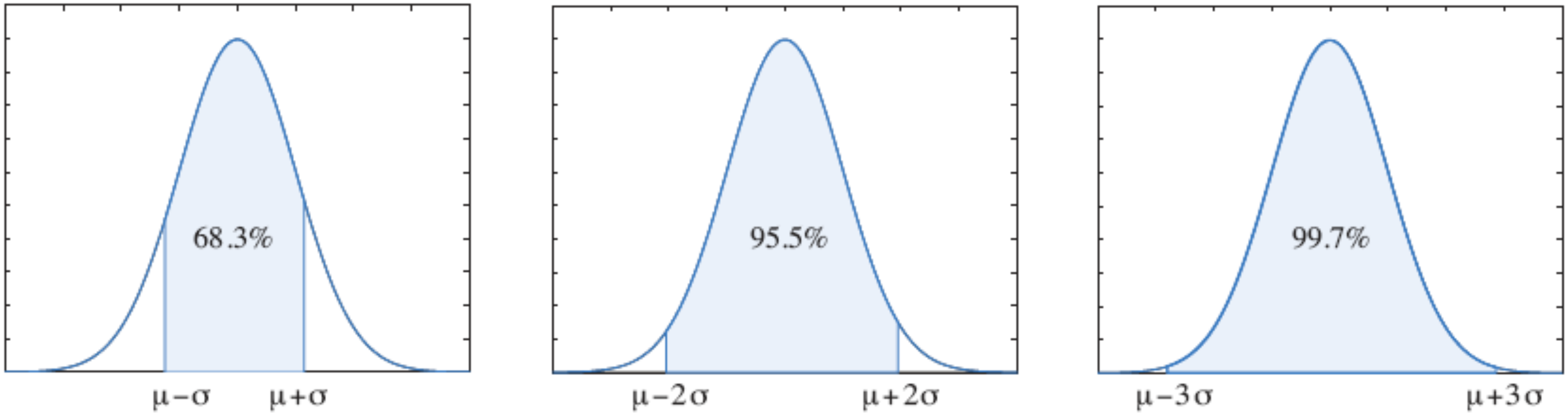
68-95-99.7 Rule

If *X* is a continuous random variable distributed normally with mean μ and standard deviation σ, then

$$P(\mu - \sigma \leq X \leq \mu + \sigma) = 0.683$$

$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 0.955$$

$$P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = 0.997$$



68-95-99.7 Rule

In words, for a normally distributed random variable:

68.3% of the values fall within one standard deviation of the mean.

95.5% of the values fall within two standard deviations of the mean.

99.7% of the values fall within three standard deviations of the mean.

Application

Example 14.7: The Lengths of Pregnancies

The lengths of human pregnancies (measured in days from conception to birth) can be approximated by the normal distribution with a mean of 266 days and a standard deviation of 16 days.

Application

Example 14.7: The Lengths of Pregnancies

Thus, about **68%** of pregnancies last between 266-16=**250 days** and 266+16=**282 days**. About **95.5%** of pregnancies last between 266-2x16=**234 days** and 266+2x16=**298 days**, and about **99.7%** of pregnancies last between 266-3x16=**218 days** and 266+3x16=**314 days**.