Basics of Probability Theory

Section 3

Elements of Set Theory

An **empty set** (denoted by $\varnothing$) is a set that contains no elements.

A set A is a **subset** of a set S, denoted by $A \subseteq S$, if A is empty or contains some or all of the elements of S.

For any set A, $\varnothing \subseteq A$, and $A \subseteq A$.

Elements of Set Theory

Definition: Intersection of Sets

The intersection $A \cap B$ of sets A and B is the set of elements that belong to both A *and* B.

Note that $A \cap B = B \cap A$.

Definition: Mutually Exclusive Events

Two events A and B are said to be mutually exclusive if they are *disjoint*, i.e., if $A \cap B = \emptyset$.

Elements of Set Theory

Definition: Union of Sets

The union $A \cup B$ of sets A and B is the set of elements that belong to either A <u>*or*</u> B.

Note that $A \cup B = B \cup A$.

Definition: Compliment of a Set

The compliment A^C of a subset $A \subseteq S$ is the set of all elements in S that are not in A.

Note that

$$S^C = \varnothing, \quad \varnothing^C = S, \quad \left(A^C\right)^C = A, \quad A \cup A^C = S, \quad A \cap A^C = \varnothing.$$

Exercise

Consider the random experiment of rolling a fair, 6-sided die. Let *A* be the event of rolling an even number and let *B* be the event of rolling a number greater than 2.

Determine the following:

(a) $A \cap B$ (b) $A \cup B$ (c) A^C

(d) A set *D* that is disjoint from both *A* and *B*.

Probability

Definition: Probability

Let S denote a sample space. A probability is a function P that assigns, to each event A in S, a unique real number P(A), called the probability of A.

The function P satisfies the following properties:

(i) $0 \le P(A) \le 1$ for any event $A \subseteq S$.

(ii) $P(\emptyset) = 0$ and $P(S) = 1$.

(iii) If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

Probability

Probability as Area:

Let the area of the sample space *S* be 1 and the area of the empty set to be 0. Any other event (subset of S) has area between 0 and 1, i.e.,

$$\text{for } A \subseteq S, \quad P(A) = \text{area of } A.$$

Exercise

Consider the random experiment of rolling a fair, 6-sided die. Let A be the event of rolling an even number and let B be the event of rolling a number greater than 2.

Determine $P(A \cap B)$.

Probability

Theorem: Probability of a Complimentary Event

If A is an event, then $P(A^C) = 1 - P(A)$.

Theorem: Probability of the Union of Two Events

If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Exercise

Consider the random experiment of rolling a fair, 6-sided die. Let A be the event of rolling an even number and let B be the event of rolling a number greater than 2.

Determine the following:

(a) $P(B^C)$ (b) $P(A \cup B)$

Assigning Probability to Events

Assume that the sample space of an experiment is finite, i.e., that it contains n distinct, simple events $E_1, E_2, \cdots, E_n$.

Then, $E_1 \cup E_2 \cup \cdots \cup E_n = S$

and $P(S) = P(E_1) + P(E_2) + \cdots + P(E_n) = 1.$

Assigning Probability to Events

Theorem: Assigning Probabilities to Equally Likely Simple Events

Assume that S is a finite sample space in which all outcomes (simple sets) are *equally likely*. The probability of an event $A \subseteq S$ is

$$P(A) = \frac{|A|}{|S|}$$

Assigning Probability to Events

Recall: Lion Population with Immigration

Consider a population of lions described by the stochastic dynamical system

$$p_{t+1} = p_t + I_t \quad \text{where} \quad I_t = \begin{cases} 12 & \text{with 50\% chance} \\ 0 & \text{with 50\% chance} \end{cases}$$

and time *t*=0,1,2,… is measured in years. If initially there are 160 lions, determine the possible population sizes 3 years later and the probability of each population size occurring.