Conditional Probability, Law of Total Probability, Bayes' Theorem

Section 4

Conditional Probability

So far, we have defined *unconditional probability*, i.e., the probability that an event has occurred, disregarding any factors that might affect it.

Conditional probability is the probability of an event occurring when we already know that another event has taken place.

Conditional Probability

Definition:

The probability of an event A *conditional* on an event C is given by

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)}$$

provided that $P(C) \neq 0.$

We read this as "the probability of A, given C".

Conditional Probability

Example #12.
A coin is tossed three times.

(a) Find the probability that at least two heads occurred.

(b) Find the probability that at least two heads occurred, given that at least one toss resulted in heads.

Conditional Probability

Notes:

(1) Conditional probability does not “commute”

(2) $P(A \cap C)$ can be calculated in two different ways

(3) $P(A | C)$ in two extreme cases

Partitions

Definition:

Let S be a sample space and $E_1, E_2, \ldots, E_n \ (n \geq 1)$ be events in S. If

(i) $E_1, E_2, \ldots, E_n$ are *mutually exclusive*, i.e.,

$$E_i \cap E_j = \varnothing \quad \text{for all } i \neq j,\ 1 \leq i, j \leq n$$

(ii) $E_1, E_2, \ldots, E_n$ are *collectively exhaustive*, i.e.,

$$S = E_1 \cup E_2 \cup \cdots \cup E_n$$

then we say that $E_1, E_2, \ldots, E_n$ form a **partition** of the sample space S.

The Law of Total Probability

Assume that events $E_1, E_2, ..., E_n \ (n \geq 1)$ form a partition of a sample space S. For any event A in S,

$$P(A) = P(A \mid E_1)P(E_1) + P(A \mid E_2)P(E_2) + \cdots + P(A \mid E_n)P(E_n)$$

The Law of Total Probability

<u>Tree Diagrams</u>

Start with the sample space S and branch it out into the events that form the partition and assign probabilities to each branch.

Create further branches and assign (this time, conditional) probabilities to them.

To find the probability of some event "A" occurring, multiply the probabilities along each path leading to A and add up these products.

Application

Incidence of Asthma in Young Adults

The incidence of asthma in young adults (assuming a 1:1 sex ratio) is 6.4% for females and 4.5% for males.

[Source: Thomsen, S.F., Ulrik, C.S., Kyvik, K.O., Larsen, K., Skadhauge, L.R., Steffensen, I., et al. (2005). The incidence of asthma in young adults. Chest, 127 (6), 1928-1934.]

Application

Incidence of Asthma in Young Adults

(a) What is the probability that a randomly chosen young adult has asthma?

(b) What is the probability that a young adult with asthma is female?

Bayes' Theorem

Assume that events $E_1, E_2, \ldots, E_n \ (n \geq 1)$ form a partition of a sample space S. Let A be an event. Then

$$P(E_i \mid A) = \frac{P(A \mid E_i)P(E_i)}{P(A \mid E_1)P(E_1) + P(A \mid E_2)P(E_2) + \cdots + P(A \mid E_n)P(E_n)}$$

Application

Example #26.
A certain medical condition (could be high blood pressure) comes in three forms, X, Y, and Z, with prevalences of 45%, 35%, and 20%, respectively. The probability that a person will need emergency medical attention is 10% if he has the X form, 5% if he has the Y form, and 45% if he has the Z form.

(a) What is the probability that a person who has the condition will require emergency medical attention?

(b) What is the probability that a person with the condition who has required emergency medical attention has the Z form?